

Risk Filtering and Risk-Averse Control of Markovian Systems Subject to Model Uncertainty

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Abstract

We consider a Markov decision process subject to model uncertainty in a Bayesian framework, where we assume that the state process is observed but its law is unknown to the observer. In addition, while the state process and the controls are observed at time t , the actual cost that may depend on the unknown parameter is not known at time t . The controller optimizes total cost by using a family of special risk measures, that we call risk filters and that are appropriately defined to take into account the model uncertainty of the controlled system. These key features lead to non-standard and non-trivial risk-averse control problems, for which we derive the Bellman principle of optimality. We illustrate the general theory on two practical examples: optimal investment and clinical trials.

1 Introduction

We study a risk-averse Markov decision problem (MDP) subject to uncertainty about the underlying dynamics as well as uncertainty about the risk-averse criterion.

Literature concerning risk-averse MDPs is rather abundant, and we refer to e.g. [FR22, FR18] and references therein. Similarly, there is a vast literature on MDPs subject to model uncertainty, and we refer to [BCC⁺19] for an overview of the classical methodologies on this topic. However, to the best of our knowledge, the present study is the first systematic study of risk-averse MDPs subject to model uncertainty. The earlier effort in [LRZ21] focuses on the CVaR criterion that has an equivalent expected value formulation. It needs to be stressed that we are not only concerned with uncertainty regarding the underlying dynamics, but also uncertainty about the optimization criterion, which is a novel and important practical feature, as two examples below show. While frequent in machine learning literature, mainly concerned with the expected value criterion, such as [LS20, SB18], it has not been addressed in the risk-averse case.

The Knightian uncertainty that we consider is parametric in nature, and our approach to tackle the respective MDP is rooted in the Bayesian methodology. It came to us as quite a surprise that accounting for possible uncertainty about the optimization criterion leads to rather intricate conceptual ideas and technical manipulations. In order to avoid measurability and integrability issues

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that are notorious and intrinsic in MDPs on general state and action spaces, and that would quite likely burden the main takeaways from this study, we decided to work with discrete state, action and parameter spaces. However, morally, the results should hold true in much more generality, that will be addressed in future works. We chose to use integral notation with respect to the state variables, which is much more pleasing to the eye and lighter than the summation notation. We keep the summation with respect to the time variable though, whenever needed.

The solution to the considered risk-averse MDP hinges on the key and new concepts of dynamic risk filters and recursive dynamic risk filters. This, in particular allows to derive a version of the dynamic programming routine suited to the needs of our uncertain risk-averse MDP.

The paper is organized as follows. In Section 2 we set the stage and define MDP and the model uncertainty framework. Also here we introduce a series of probability measures and some of their properties used frequently in the sequel. Section 3 is devoted to risk filters, starting with the definition and some fundamental properties of these objects. The key concept of parameter consistency of risk filters is introduced in Section 3.2, while the time consistency of risk filters is studied in Section 3.3. In this section we provide a characterization of parameter consistent and time consistent risk filters; cf. Theorem 3.14. Also here we discuss two important examples of risk filters: expectation of an additive functional, Example 3.15, and risk-sensitive criteria in the context of clinical trials, Example 3.6. The structure Theorem 3.14 leads to the notation of recursive risk filters, introduced in Section 4. Section 5.3 is devoted to the risk-averse control problem. In Section 5.1 we derive the Bayes operator for the posterior of the parameter of interest. Then, we derive the dynamic programming backward recursion for the classical additive reward case; Section 5.2. Here, as a particular case, we briefly discuss the optimal investment and consumption problem, when the investor faces Knightian uncertainty and unknown risk-aversion parameter; Example 5.4. We conclude with the solution to the optimal control problem for a general recursive risk filter: Theorem 5.6.

Finally, we want to mention that while writing this manuscript we strove to keep a balance between heavy notations and rigor. Nevertheless, some formulas still may appear overwhelming, which is typically the case for MDPs.

2 Markov Decision Processes with Model Uncertainty

We consider an observed, controlled random process $X = \{X_t\}_{t=1,\dots,T}$. The corresponding state space is a finite set \mathcal{X} . The underlying probability space that we will work with is canonical. It includes the space of paths of X : $\Omega = \underbrace{\mathcal{X} \times \dots \times \mathcal{X}}_{T \text{ times}} = (\mathcal{X})^T$, endowed with the canonical product σ -

field $\mathcal{F} = \underbrace{2^{\mathcal{X}} \otimes \dots \otimes 2^{\mathcal{X}}}_{T \text{ times}}$. The elements of Ω are $\omega = (\omega_1, \dots, \omega_T)$. We use x_t to denote the canonical

projections at time t , so that $X_t(\omega) = x_t = \omega_t$. We let $\{\mathcal{F}_t^X\}_{t=1,\dots,T}$ to denote the canonical filtration generated by the process X , so that $\mathcal{F}_t^X = \underbrace{2^{\mathcal{X}} \otimes \dots \otimes 2^{\mathcal{X}}}_{t \text{ times}} \otimes \underbrace{\{\Omega, \emptyset\} \otimes \dots \otimes \{\Omega, \emptyset\}}_{T-t \text{ times}}$. We will make use

of the notations $\mathcal{T} = \{1, \dots, T\}$ and $\mathcal{T}_t = \{t, \dots, T\}$.

The control space is given by a finite set \mathcal{U} , and the set of admissible controls at step t is given by a multifunction $\mathcal{U}_t : \mathcal{X} \rightrightarrows \mathcal{U}$ with nonempty values. We consider a parametric family of transition kernels $K_\theta : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{P}(\mathcal{X})$, where $\mathcal{P}(\mathcal{X})$ is the space of probability measures on \mathcal{X} , and $\theta \in \Theta$ represents an unknown parameter. Here, Θ is a finite set. The unknown true value of the parameter θ is θ^* .

We will consider the Bayesian setting, and therefore, we consider the product space $\widehat{\Omega} = \Omega \times \Theta$ endowed with product σ -algebra $\widehat{\mathcal{F}} := \mathcal{F} \otimes 2^\Theta$. We denote $\widehat{\omega} = (\omega, \theta)$ and $\widehat{\omega}_t = (\omega_t, \theta)$. In accordance with the Bayesian setting we denote by Θ a random variable on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ with values in Θ , and with

$\Theta(\widehat{\omega}) = \theta$. We also assume that some *prior distribution* ξ_1 of Θ (supported in Θ) is available.

The process $\{X_t\}_{t \in \mathcal{T}}$ considered as a process on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ is denoted as $\widehat{X} = \{\widehat{X}_t\}_{t \in \mathcal{T}}$, and $\widehat{X}_t(\widehat{\omega}) = X_t(\omega)$. Accordingly, the canonical filtration generated by the process \widehat{X} is given as $\{\widehat{\mathcal{F}}_t^{\widehat{X}} = \mathcal{F}_t^X \otimes 2^\Theta, t \in \mathcal{T}\}$.

At time t , the history of observed states is $h_t = (x_1, x_2, \dots, x_t)$, while all the information available for making a decision is $g_t = (x_1, u_1, x_2, u_2, \dots, x_t)$. We use $\mathcal{H}_t := \mathcal{X}^t = \underbrace{\mathcal{X} \times \dots \times \mathcal{X}}_{t \text{ times}}$ to denote the

spaces of possible state histories h_t . We make distinction of g_t and h_t because we should make decision of u_t based on g_t as the past controls u_1, \dots, u_{t-1} are also taken into consideration when estimating the conditional distribution of θ . We write H_t for (X_1, \dots, X_t) and \widehat{H}_t for $(\widehat{X}_1, \dots, \widehat{X}_t)$.

A *history-dependent admissible policy* $\pi = (\pi_1, \dots, \pi_T)$ is a sequence of functions $\pi_t(g_t)$ such that $\pi_t(g_t) \in \mathcal{U}_t(x_t)$ for all possible g_t . One can easily prove that for such an admissible policy π , each π_t reduces to a function of $h_t = (x_1, x_2, \dots, x_t)$,¹ as $u_s = \pi_s(x_1, \dots, x_s)$ for all $s = 1, \dots, t-1$. Therefore the set of admissible policies is

$$\Pi = \left\{ \pi = (\pi_1, \dots, \pi_T) : \pi_t(x_1, \dots, x_t) \in \mathcal{U}_t(x_t), t \in \mathcal{T} \right\}.$$

Any policy $\pi \in \Pi$ defines the control process, also denoted by $\pi = \{\pi_t\}_{t \in \mathcal{T}}$, with $\pi_t = \pi_t(X_1, \dots, X_t)$. We make a distinction between $u_t = \pi_t(x_1, \dots, x_t)$ and $\pi_t = \pi_t(X_1, \dots, X_t)$.

As said in the Introduction, even though we work with discrete spaces \mathcal{X} and Θ , we are using the more convenient integral notation, rather than the summation notation.

For a fixed initial state x_1 , every policy $\pi \in \Pi$, and every $\theta \in \Theta$, a probability measure P_θ^π on (Ω, \mathcal{F}) is uniquely defined by:

$$\begin{aligned} P_\theta^\pi(A_1 \times A_2 \times \dots \times A_{T-1} \times A_T) &= \int_{A_1} \int_{A_2} \dots \int_{A_{T-1}} K_\theta(A_T | x_{T-1}, \pi_{T-1}(x_1, \dots, x_{T-1})) \\ &\quad \times K_\theta(dx_{T-1} | x_{T-2}, \pi_{T-2}(x_1, \dots, x_{T-2})) \times \dots \\ &\quad \dots \times K_\theta(dx_2 | x_1, \pi_1(x_1)) \delta_{x_1}(dy), \quad A_t \subset \mathcal{X}, t \in \mathcal{T}, \end{aligned} \quad (2.1)$$

where, as usual, δ_x denotes the Dirac measure concentrated at x . In particular,

$$P_\theta^\pi(A) = P_\theta^\pi(X \in A) = P_\theta^\pi(\{\omega \in \Omega : X(\omega) \in A\}), \quad A \subset \Omega.$$

The true but unknown measure under the policy π is $P_{\theta^*}^\pi$. This measure gives the true law of the canonical process X subject to control strategy π .

Given the prior distribution ξ_1 , a probability measure P^π on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ is defined as well:

$$P^\pi(A \times D) = \int_D P_\theta^\pi(A) \xi_1(d\theta), \quad A \subset \Omega, D \subset \Theta. \quad (2.2)$$

In particular,

$$P^\pi(A \times D) = P^\pi(\{\widehat{\omega} \in \widehat{\Omega} : \widehat{X}(\widehat{\omega}) \in A, \Theta(\widehat{\omega}) \in D\}).$$

Clearly, ξ_1 is the marginal of P^π , that is $\xi_1(D) = P^\pi(\Omega \times D)$. To simplify the ensuing study, we assume that for any $t \in \mathcal{T}$ and $h_t \in \mathcal{H}_t$ we have $P^\pi(\widehat{H}_t = h_t) > 0$. This assumption is of course an assumption about the kernels K_θ , $\theta \in \Theta$.

Furthermore, for each $t = 1, \dots, T-1$ and for each history $h_t \in \mathcal{H}_t$, we define the set of *tail control strategies*

$$\Pi^{t, h_t} = \left\{ \pi^{t, h_t} : \pi_t^{t, h_t} = \pi_t(h_t), \pi_s^{t, h_t}(x_{t+1}, \dots, x_s) = \pi_s(h_t, x_{t+1}, \dots, x_s), s \in \mathcal{T}_{t+1}, \pi \in \Pi \right\}.$$

¹We are still using π_s to denote the decision rule; it will not lead to any misunderstanding.

In addition, for $t = 1, \dots, T-1$, and for each $\theta \in \Theta$, $h_t \in \mathcal{H}_t$, and $\pi^{t,h_t} \in \Pi^{t,h_t}$ we construct a probability measure $P_{\theta,t+1,T}^{\pi^{t,h_t}}$ on \mathcal{X}^{T-t} in analogy to (2.1). Specifically, we put

$$\begin{aligned} P_{\theta,t+1,T}^{\pi^{t,h_t}}(A_{t+1} \times \dots \times A_T) &= \int_{A_{t+1}} \int_{A_{t+2}} \dots \int_{A_{T-1}} K_{\theta}(A_T | x_{T-1}, \pi_{T-1}(h_t, x_{t+1}, \dots, x_{T-1})) \\ &\quad \cdot K_{\theta}(dx_{T-1} | x_{T-2}, \pi_{T-2}(h_t, x_{t+1}, \dots, x_{T-2})) \dots K_{\theta}(dx_{t+2} | x_{t+1}, \pi_{t+1}(h_t, x_{t+1})) \\ &\quad \cdot K_{\theta}(dx_{t+1} | x_t, \pi_t(h_t)), \quad A_s \subset \mathcal{X}, \quad s \in \mathcal{T}_{t+1}. \end{aligned} \quad (2.3)$$

We proceed with three technical results that are rather straightforward consequences of the above set-up.

Lemma 2.1. *For any $h_t \in \mathcal{H}_t$, and $A_s \subset \mathcal{X}$, $s \in \mathcal{T}_{t+1}$, $\pi \in \Pi$, and the corresponding $\pi^{t,h_t} \in \Pi^{t,h_t}$ we have that*

$$P_{\theta,t+1,T}^{\pi^{t,h_t}}(A_{t+1} \times \dots \times A_T) = P_{\theta}^{\pi}(X_{t+1} \in A_{t+1}, \dots, X_T \in A_T | H_t = h_t). \quad (2.4)$$

Proof. First, note that²

$$P_{\theta}^{\pi}(X_1 = x_1, \dots, X_t = x_t) = K_{\theta}(x_2 | x_1, \pi_1(x_1)) K_{\theta}(x_3 | x_2, \pi_2(x_1, x_2)) \dots K_{\theta}(x_t | x_{t-1}, \pi_{t-1}(x_1, \dots, x_t)).$$

On the other hand,

$$\begin{aligned} P_{\theta}^{\pi}(X_{t+1} \in A_{t+1}, \dots, X_T \in A_T, H_t = h_t) &= P_{\theta}^{\pi}(X_1 = x_1, \dots, X_t = x_t, X_{t+1} \in A_{t+1}, \dots, X_T \in A_T) \\ &= \int_{\{x_1\}} \dots \int_{\{x_t\}} P_{\theta,t+1,T}^{\pi^{t,\bar{h}_t}}(A_{t+1} \times \dots \times A_T) K_{\theta}(d\bar{x}_t | \bar{x}_{t-1}, \pi_{t-1}(\bar{h}_{t-1})) \dots K_{\theta}(d\bar{x}_2 | \bar{x}_1, \pi_1(\bar{x}_1)) \delta_{x_1}(y) \\ &= P_{\theta,t+1,T}^{\pi^{t,h_t}}(A_{t+1} \times \dots \times A_T) K_{\theta}(x_t | x_{t-1}, \pi_{t-1}(h_{t-1})) \dots K_{\theta}(x_2 | x_1, \pi_1(x_1)). \end{aligned}$$

Combining the above we immediately have (2.4). \square

For future reference we denote by $P_{\theta,t+1}^{\pi^{t,h_t}}$ a measure on $(\mathcal{X}, 2^{\mathcal{X}})$ defined as

$$P_{\theta,t+1}^{\pi^{t,h_t}}(B) = P_{\theta,t+1,T}^{\pi^{t,h_t}}(B \times \mathcal{X}^{T-t-1}) = P_{\theta,t+1,T}^{\pi}(X_{t+1} \in B). \quad (2.5)$$

Thus, we have that, for $t \leq T-1$,

$$P_{\theta,t+1}^{\pi^{t,h_t}}(B) = \int_B K_{\theta}(dx_{t+1} | x_t, \pi_t(h_t)) = K_{\theta}(B | x_t, \pi_t(h_t)) = P_{\theta}^{\pi}(X_{t+1} \in B | H_t = h_t). \quad (2.6)$$

Next, we construct a probability measure $P_{t+1,T}^{\pi^{t,h_t}}$ on $\mathcal{X}^{T-t} \times \Theta$ as

$$P_{t+1,T}^{\pi^{t,h_t}}(A \times D) = \int_D P_{\theta,t+1,T}^{\pi^{t,h_t}}(A) \xi_t^{\pi,h_t}(d\theta), \quad A \in \underbrace{2^{\mathcal{X}} \otimes \dots \otimes 2^{\mathcal{X}}}_{T-t \text{ times}}, \quad D \in 2^{\Theta}, \quad (2.7)$$

where $\xi_t^{\pi,h_t} \in \mathcal{P}(\Theta)$, is given as

$$\xi_t^{\pi,h_t}(D) = P^{\pi}(\Theta \in D | \widehat{H}_t = h_t), \quad \text{for } t = 2, \dots, T, \quad \text{and} \quad \xi_1^{\pi,h_1}(D) = \xi_1(D). \quad (2.8)$$

We note that we have the following identity for the conditional measure

$$P_{t+1,T}^{\pi^{t,h_t}}(A | \Theta = \theta) = P_{\theta,t+1,T}^{\pi^{t,h_t}}(A), \quad \theta \in \Theta, \quad A \in 2^{\mathcal{X}} \otimes \dots \otimes 2^{\mathcal{X}}. \quad (2.9)$$

²To further simplify the notation we write $K_{\theta}(x|\dots)$ in place of $K_{\theta}(\{x\}|\dots)$. In a similar way, for a probability measure Q on \mathcal{Q} , and $y \in \mathcal{Q}$, we may write $Q(y)$ instead of $Q(\{y\})$.

Lemma 2.2. Let $A \in \underbrace{2^{\mathcal{X}} \otimes \cdots \otimes 2^{\mathcal{X}}}_{T-t \text{ times}}$, $D \in 2^{\Theta}$, $h_t \in \mathcal{H}_t$, and $\pi \in \Pi$. Then:

(i) We have

$$P_{t+1,T}^{\pi^{t,h_t}}(A \times D) = P^\pi((\widehat{X}_{t+1}, \widehat{X}_{t+2}, \dots, \widehat{X}_{T-1}, \widehat{X}_T) \in A, \Theta \in D | \widehat{H}_t = h_t). \quad (2.10)$$

Proof. First, in view of (2.8) and (2.2) we note that

$$\xi_t^{\pi, h_t}(D) = \frac{\int_D P_\theta^\pi(H_t = h_t) \xi_1(d\theta)}{P^\pi(\widehat{H}_t = h_t)}.$$

Thus,

$$P_{t+1,T}^{\pi^{t,h_t}}(A \times D) = \frac{\int_D P_{\theta,t+1,T}^{\pi^{t,h_t}}(A) P_\theta^\pi(H_t = h_t) \xi_1(d\theta)}{P^\pi(\widehat{H}_t = h_t)}. \quad (2.11)$$

On the other hand, using (2.2) and Lemma 2.1, we have

$$\begin{aligned} & P^\pi((\widehat{X}_{t+1}, \widehat{X}_{t+2}, \dots, \widehat{X}_{T-1}, \widehat{X}_T) \in A, \Theta \in D | \widehat{H}_t = h_t) \\ &= \frac{\int_D P_\theta^\pi((X_{t+1}, \dots, X_T) \in A, H_t = h_t) \xi_1(d\theta)}{P^\pi(\widehat{H}_t = h_t)} \\ &= \frac{\int_D P_\theta^\pi((X_{t+1}, \dots, X_T) \in A | H_t = h_t) P_\theta^\pi(H_t = h_t) \xi_1(d\theta)}{P^\pi(\widehat{H}_t = h_t)} \\ &= \frac{\int_D P_{\theta,t+1}^{\pi^{t,h_t}}(A) P_\theta^\pi(H_t = h_t) \xi_1(d\theta)}{P^\pi(\widehat{H}_t = h_t)}. \end{aligned}$$

This, combined with (2.11) concludes the proof of part (i). □

Remark 2.3. Formally, taking $T = t + 1$ in (2.9) we obtain

$$P_{t+1,t+1|\theta}^{\pi^{t,h_t}}(A) = P_{\theta,t+1,t+1}^{\pi^{t,h_t}}(A) = P_{\theta,t+1}^{\pi^{t,h_t}}(A), \quad (2.12)$$

where in the last equality we used (2.5).

Lemma 2.4. Let $t \in \{1, \dots, T-1\}$, and let F be a function on $\mathcal{X}^{T-t} \times \Theta$. Then, for each $h_t \in \mathcal{H}_t$ we have

$$\begin{aligned} & E^\pi[F(\widehat{X}_{t+1}, \widehat{X}_{t+2}, \dots, \widehat{X}_{T-1}, \widehat{X}_T, \Theta) | \widehat{H}_t = h_t] = \\ & \int_{\Theta} \int_{\mathcal{X}^{T-t}} F(x_{t+1}, \dots, x_T, \theta) P_{\theta,T}^{\pi^{t,h_t}}(dx_T) \cdots P_{\theta,t+1}^{\pi^{t,h_t}}(dx_{t+1}) \xi_t^{\pi, h_t}(d\theta), \end{aligned} \quad (2.13)$$

where E^π denotes the expectation with respect to probability P^π .

Proof. In view of Lemma 2.2, we have

$$P^\pi(dx_{t+1}, \dots, dx_T; d\theta | \widehat{H}_t = h_t) = P_{t+1,T}^{\pi^{t,h_t}}(dx_{t+1}, \dots, dx_T; d\theta).$$

Consequently, by (2.7), we continue

$$P_{t+1,T}^{\pi^{t,h_t}}(dx_{t+1}, \dots, dx_T; d\theta) = P_{\theta,t+1,T}^{\pi^{t,h_t}}(dx_{t+1}, \dots, dx_T) \xi_t^{\pi^{t,h_t}}(d\theta).$$

This combined with (2.3) and (2.6) yields the identity (2.13). □

For future reference we denote by $P_{t+1}^{\pi^{t,h_t}}$ the measure on $\mathcal{X} \times \Theta$ defined as

$$P_{t+1}^{\pi^{t,h_t}}(B \times D) = P_{t+1,T}^{\pi^{t,h_t}}(B \times \mathcal{X}^{T-t-1} \times D) = P^\pi(\widehat{X}_{t+1} \in B, \Theta \in D \mid \widehat{H}_t = h_t), \quad (2.14)$$

for $t = 1, \dots, T-1$, with the convention, employed throughout, that $B \times \mathcal{X}^0 \times D = B \times D$.

For $t = T$ and $h_T \in \mathcal{H}_T$ we construct a measure $P_{T+1,T}^{\pi^{T,h_T}}$ on Θ as

$$P_{T+1,T}^{\pi^{T,h_T}}(D) = P^\pi(\Theta \in D \mid \widehat{H}_T = h_T) = \xi_T^{\pi,h_T}(D).$$

Given that a strategy π is used, then at each time $t \in \mathcal{T}$ a random cost $Z_{\theta^*,t}^\pi$ is incurred, with

$$Z_{\theta,t}^\pi = c_t(X_t, \pi_t, \theta),$$

where $c_t : \mathcal{X} \times \mathcal{U} \times \Theta \rightarrow \mathbb{R}_+$.

Remark 2.5. It is important to note that even though X_t and π_t are observed at time t , the actual cost $c_t(X_t, \pi_t, \theta^*)$ is not known (or observed) at time t as θ^* is not known. The dependence of both the transition kernel and the accrued costs on the unknown parameter is an important practical situation, leading to non-standard and non-trivial risk-averse Markov decision problems.

To proceed, for each $t \in \mathcal{T}$ and each history $h_t \in \mathcal{H}_t$ we denote

$$Z_{\theta,t,t}^{\pi,h_t} = c_t(x_t, \pi_t(h_t), \theta), \quad (2.15)$$

and for each $s = t+1, \dots, T$ we put

$$Z_{\theta,t,s}^{\pi,h_t,x_{t+1},\dots,x_s} = c_s(x_s, \pi_s^{t,h_t}(x_{t+1}, \dots, x_s), \theta). \quad (2.16)$$

Note that, for a fixed strategy π and a fixed $h_t \in \mathcal{H}_t$, we have that $c_t(x_t, \pi_t(h_t), \cdot)$ is a function on Θ , and $c_s(\cdot, \pi_s^{t,h_t}(\cdot, \dots, \cdot), \theta)$ is a function on $\mathcal{X}^{s-t} \times \Theta$.

3 Risk Filters for MDPs with Model Uncertainty

3.1 Dynamic risk filters

For $t = 1, \dots, T-1$ and $s = t, \dots, T$, we denote by $\mathcal{Z}_t^{\mathcal{X}}$ and $\mathcal{Z}_{t,s}$ the spaces of real valued functions on \mathcal{X}^t and $\mathcal{X}^{s-t} \times \Theta$, respectively, where $\mathcal{X}^0 \times \Theta := \Theta$, so that $\mathcal{Z}_{t,t}$ is the space of real valued functions on Θ . For $Z_{t,s}, W_{t,s} \in \mathcal{Z}_{t,s}$, the comparison between these functions is understood point-wise; $Z_{t,s} \leq W_{t,s}$ means that $Z_{t,s}(x_{t+1}, \dots, x_s, \theta) \leq W_{t,s}(x_{t+1}, \dots, x_s, \theta)$ for all $(x_{t+1}, \dots, x_s, \theta) \in \mathcal{X}^{s-t} \times \Theta$.

For any policy $\pi \in \Pi$, our objective is to evaluate at each time $t \in \mathcal{T}$, the riskiness of the sequence of costs $Z_{\Theta,t}^{\pi,h_t}, Z_{\Theta,t,t+1}^{\pi,h_t,X_{t+1}}, \dots, Z_{\Theta,t,T}^{\pi,h_t,X_{t+1},\dots,X_T}$, given history h_t , in such a way that the evaluation is $\mathcal{F}_t^{\mathcal{X}}$ -measurable. We denote by

$$\mathcal{Z}^{t,T} = \mathcal{Z}_{t,t} \times \mathcal{Z}_{t,t+1} \times \dots \times \mathcal{Z}_{t,T}$$

the space of *conditional cost functions*³ in periods t, \dots, T .

For $t = 1, \dots, T$ and $s = t, \dots, T$ we also use $\mathcal{P}_{t,s}$ to denote the space of probability measures on the space of paths starting at time t and ending at time s , and on realizations of the parameter θ , that is on the space $\mathcal{X}^{s-t+1} \times \Theta$. Additionally, we understand $\mathcal{P}_{T+1,T}$ as $\mathcal{P}(\Theta)$, because no future paths are possible. Note, in particular, that $P_{t+1,T}^{\pi^{t,h_t}} \in \mathcal{P}_{t+1,T}$, for $t \in \mathcal{T}$, and $P_{t+1,t+1}^{\pi^{t,h_t}} \in \mathcal{P}_{t+1,t+1}$.

Observe that at time $t = 1, \dots, T-1$ we know the history h_t , and, for any policy $\pi \in \Pi$ (in principle), we can evaluate the distribution of $(X_{t+1}, \dots, X_T, \Theta)$ under the measure $P_{t+1,T}^{\pi^{t,h_t}}$.

We proceed with stating three key definitions.

³The term *conditional* refers to the fact that at any time t we consider cost functions that depend on a history h_t .

Definition 3.1. For a fixed $t \in \mathcal{T}$, a mapping $\rho_t : \mathcal{Z}^{t,T} \times \mathcal{P}_{t+1,T} \rightarrow \mathbb{R}$, is called a *conditional risk filter*.

Note that, in particular, for any $(Z_{t,t}, \dots, Z_{t,T}) \in \mathcal{Z}^{t,T}$ and $\pi \in \Pi$ we have

$$\rho_t(Z_{t,t}, \dots, Z_{t,T}; P_{t+1,T}) = R(h_t), \text{ for all } h_t \in \mathcal{X}^t,$$

for some function $R : \mathcal{X}^t \rightarrow \mathbb{R}$.

Definition 3.2. Let $t \in \mathcal{T}$. A conditional risk filter ρ_t

- (i) is *normalized* if $\rho_t(0, 0, \dots, 0; P_{t+1,T}) = 0$ for all $P_{t+1,T} \in \mathcal{P}_{t+1,T}$;
- (ii) is *monotonic* if $\rho_t(Z_{t,t}, \dots, Z_{t,T}; P_{t+1,T}) \leq \rho_t(W_{t,t}, \dots, W_{t,T}; P_{t+1,T})$ for all $P_{t+1,T} \in \mathcal{P}_{t+1,T}$, and all $(Z_{t,t}, \dots, Z_{t,T})$ and $(W_{t,t}, \dots, W_{t,T})$ in $\mathcal{Z}^{t,T}$, such that $Z_{t,s} \leq W_{t,s}$ for all $s \in \mathcal{T}_t$;
- (iii) is *translation invariant* if for all $(Z_{t,t}, \dots, Z_{t,T}) \in \mathcal{Z}^{t,T}$, all $V \in \mathbb{R}$, and all $P_{t+1,T} \in \mathcal{P}_{t+1,T}$,

$$\rho_t(V + Z_{t,t}, Z_{t,t+1}, \dots, Z_{t,T}; P_{t+1,T}) = V + \rho_t(Z_{t,t}, Z_{t,t+1}, \dots, Z_{t,T}; P_{t+1,T});$$

- (iv) has the *support property*, if

$$\rho_t(Z_{t,t}, \dots, Z_{t,T}; P_{t+1,T}) = \rho_t(Z_{t,t} \mathbb{1}_{\text{supp}_t(P_{t+1,T})}, \dots, Z_{t,T} \mathbb{1}_{\text{supp}_T(P_{t+1,T})}; P_{t+1,T}),$$

for all $(Z_{t,t}, \dots, Z_{t,T}) \in \mathcal{Z}^{t,T}$, and all $P_{t+1,T} \in \mathcal{P}_{t+1,T}$, and where $\text{supp}_s(P_{t+1,T})$ denotes the projection of $\text{supp}(P_{t+1,T})$ on $\mathcal{X}^{s-t} \times \Theta$, for $s \geq t$.

Remark 3.3. Let $s = t, \dots, T$ and let $\{Z_{s,T}^y, y \in \mathcal{Y}\}$ be a family of functions parameterized by y , for some non-empty set \mathcal{Y} . Then, by the normalization property, for any $A \subset \mathcal{Y}$, $y \in \mathcal{Y}$, and $P \in \mathcal{P}_{t+1,T}$, we have that

$$\mathbb{1}_A(y) \rho_t(Z_{t,T}^y, \dots, Z_{T,T}^y; P) = \rho_t(\mathbb{1}_A(y) Z_{t,T}^y, \dots, \mathbb{1}_A(y) Z_{T,T}^y; P).$$

Definition 3.4. A *dynamic risk filter* $\rho = \{\rho_t\}_{t \in \mathcal{T}}$ is a sequence of conditional risk filters $\rho_t : \mathcal{Z}^{t,T} \times \mathcal{P}_{t+1,T} \rightarrow \mathbb{R}$. We say that it is normalized, monotonic, translation invariant, or has the support property, if all ρ_t , $t \in \mathcal{T}$, satisfy the respective conditions of Definition 3.2.

3.2 Parameter Consistency

Let $t = 1, \dots, T-1$ and $s = t, \dots, T$. For any probability measure $P_{t,s} \in \mathcal{P}_{t,s}$, we denote by $P_{t,s|\Theta}(\cdot, \cdot)$, the stochastic kernel from Θ to \mathcal{X}^{s-t+1} defined as

$$P_{t,s|\Theta}(\theta, A) = \frac{P_{t,s}(A \times \{\theta\})}{P_{t,s}(\mathcal{X}^{s-t+1} \times \{\theta\})}, \quad A \subset \mathcal{X}^{s-t+1}, \theta \in \Theta.$$

The corresponding marginal on Θ is denoted by $P_{t,s,\Theta}$, so that

$$P_{t,s,\Theta}(D) = P_{t,s}(\mathcal{X}^{s-t+1} \times D), \quad D \subset \Theta.$$

Clearly, the measure $P_{t,s}$ admits disintegration

$$P_{t,s}(A \times B) = \int_B P_{t,s|\Theta}(A) P_{t,s,\Theta}(d\theta) =: P_{t,s,\Theta} \otimes P_{t,s|\Theta}(A \times B),$$

where we use a simplified notation

$$P_{t,s|\Theta}(A) := P_{t,s|\Theta}(\theta, A).$$

We note that for any stochastic kernel $\kappa_{s,t}(\cdot, \cdot)$ from Θ to \mathcal{X}^{s-t} and for any probability measure μ on 2^Θ one can construct a unique probability measure on the product space $\mathcal{X}^{s-t+1} \times \Theta$ as

$$m_{t,s}(A \times B) = \int_B \kappa_{t,s}(\theta, A) \mu(d\theta) =: \mu \otimes \kappa_{t,s}(A \times B).$$

In particular, with $\mu = \delta_\theta$ and $\kappa_{t,s} = P_{t,s|\Theta}$, with $P_{t,s} \in \mathcal{P}_{t,s}$, we get

$$m_{t,s}(A \times B) = \delta_\theta \otimes P_{t,s|\Theta}(A \times B) = P_{t,s|\theta}(A) \mathbb{1}_B(\theta) = P_{t,s|\theta}(A) \delta_\theta(B). \quad (3.1)$$

Remark 3.5. In our convention, $P_{T+1,T}(\cdot)$ is a measure on Θ . This means that, formally, $P_{T+1,T,\Theta} = P_{T+1,T}$ and $P_{T+1,T|\Theta} \equiv 1$, in which case (formally)

$$\delta_\theta \otimes P_{T+1,T|\Theta} = \delta_\theta.$$

Example 3.6. Fix $t \in \mathcal{T}$, $h_t \in \mathcal{H}_t$ and $\pi \in \Pi$. Take $P_{t+1,T} = P_{t+1,T}^{\pi^{t,h_t}} \in \mathcal{P}_{t+1,T}$ and $P_{t+1,t+1} = P_{t+1,t+1}^{\pi^{t,h_t}} \in \mathcal{P}_{t+1,t+1}$. Then,

$$P_{t+1,T|\theta} = P_{\theta,t+1,T}^{\pi^{t,h_t}}, \quad P_{t+1,t+1|\theta} = P_{\theta,t+1}^{\pi^{t,h_t}}, \quad P_{t+1,T,\Theta} = \xi_t^{\pi^{t,h_t}}. \quad (3.2)$$

The first equality above comes from (2.9). The second one comes from (2.12). The third one is just (2.7) with $A = \mathcal{X}^{T-t}$. Note that (3.1) and (3.2) imply that

$$\delta_\theta \otimes P_{t+1,T|\Theta}^{\pi^{t,h_t}}(A \times B) = \delta_\theta \otimes P_{\theta,t+1,T}^{\pi^{t,h_t}}(A \times B), \quad \delta_\theta \otimes P_{t+1,t+1|\Theta}^{\pi^{t,h_t}}(A \times B) = \delta_\theta \otimes P_{\theta,t+1,t+1}^{\pi^{t,h_t}}(A \times B). \quad (3.3)$$

We introduce the following key concept.

Definition 3.7. A conditional risk filter $\rho_t : \mathcal{Z}^{t,T} \times \mathcal{P}_{t+1,T} \rightarrow \mathbb{R}$ is *parameter consistent*, if for all $(Z_{t,t}, \dots, Z_{t,T})$, $(W_{t,t}, \dots, W_{t,T}) \in \mathcal{Z}^{t,T}$, and all $P_{t+1,T}, Q_{t+1,T} \in \mathcal{P}_{t+1,T}$ the relations

$$P_{t+1,T,\Theta} = Q_{t+1,T,\Theta}$$

and

$$\rho_t(Z_{t,t}, \dots, Z_{t,T}; \delta_\theta \otimes P_{t+1,T|\Theta}) \leq \rho_t(W_{t,t}, \dots, W_{t,T}; \delta_\theta \otimes Q_{t+1,T|\Theta}), \quad \text{for all } \theta \in \Theta, \quad (3.4)$$

imply that

$$\rho_t(Z_{t,t}, \dots, Z_{t,T}; P_{t+1,T}) \leq \rho_t(W_{t,t}, \dots, W_{t,T}; Q_{t+1,T}). \quad (3.5)$$

In words, if the marginal distributions of P and Q on Θ are the same, and the conditional risk of $Z_{t:T} := (Z_{t,t}, \dots, Z_{t,T})$ under P is not greater than that of $W_{t:T} := (W_{t,t}, \dots, W_{t,T})$ under Q for every value of θ , then the risk of $Z_{t:T}$ under P should be not greater than that of $W_{t:T}$ under Q .

Remark 3.8. Note that parameter consistency at $t = T$ follows from the support property, translation invariance, monotonicity, and normalization of ρ_T . Indeed, first observe that according to Remark 3.5 the equality $P_{T+1,T,\Theta} = Q_{T+1,T,\Theta} = 1$ implies that $P_{T+1,T} = Q_{T+1,T}$. Thus, for any $\theta \in \Theta$

$$\begin{aligned} \rho_T(Z_{T,T}, \delta_\theta) \leq \rho_T(W_{T,T}, \delta_\theta) &\Leftrightarrow \rho_T(Z_{T,T}(\theta), \delta_\theta) \leq \rho_T(W_{T,T}(\theta), \delta_\theta) \Leftrightarrow \\ Z_{T,T}(\theta) + \rho_T(0, \delta_\theta) \leq W_{T,T}(\theta) + \rho_T(0, \delta_\theta) &\Leftrightarrow Z_{T,T}(\theta) \leq W_{T,T}(\theta). \end{aligned}$$

By monotonicity, we have that

$$\rho_T(Z_{T,T}, P_{T+1,T}) \leq \rho_T(W_{T,T}, P_{T+1,T}) = \rho_T(W_{T,T}, Q_{T+1,T}).$$

This remark is used in Proposition 3.10 and also in Theorem 3.14.

We have the following risk decomposition formula.

Theorem 3.9. *Take $t = 1, \dots, T$. If a conditional risk filter $\rho_t : \mathcal{Z}^{t,T} \times \mathcal{P}_{t+1,T} \rightarrow \mathbb{R}$ is parameter consistent, then there exists a mapping $\hat{\rho}_t : \mathcal{Z}_{t,t} \times \mathcal{P}(\Theta) \rightarrow \mathbb{R}$ such that for all $Z_{t:T}$ and $P_{t+1,T}$,*

$$\rho_t(Z_{t,t}, \dots, Z_{t,T}; P_{t+1,T}) = \hat{\rho}_t\left(\{\rho_t(Z_{t,t}, \dots, Z_{t,T}; \delta_\theta \otimes P_{t+1,T|\Theta}), \theta \in \Theta\}; P_{t+1,T,\Theta}\right). \quad (3.6)$$

Proof. Suppose two sequences $Z_{t:T}$ and $W_{t:T}$ in $\mathcal{Z}^{t,T}$, and two measures $P_{t+1,T}$ and $Q_{t+1,T}$ in $\mathcal{P}_{t+1,T}$ are such that $P_{t+1,T,\Theta} = Q_{t+1,T,\Theta}$ and

$$\rho_t(Z_{t,t}, \dots, Z_{t,T}; \delta_\theta \otimes P_{t+1,T|\Theta}) = \rho_t(W_{t,t}, \dots, W_{t,T}; \delta_\theta \otimes Q_{t+1,T|\Theta}), \quad \forall \theta \in \Theta.$$

Then it follows from Definition 3.7 that

$$\rho_t(Z_{t,t}, \dots, Z_{t,T}; P_{t+1,T}) = \rho_t(W_{t,t}, \dots, W_{t,T}; Q_{t+1,T}).$$

This means that formula (3.6) is true. \square

Thus, parameter consistency allows us to disintegrate the risk filtering task into two stages. First, we evaluate the risk in a fully observed system, with the parameter θ fixed, and then we integrate the results by using the operator $\hat{\rho}_t$, which we call the *marginal risk filter*.

Proposition 3.10. *Take $t = 1, \dots, T$. If the conditional risk filter ρ_t is parameter consistent, normalized, monotonic, and has the translation invariant property, or the support property, then the mapping $\hat{\rho}_t(\cdot; \cdot)$ has the corresponding properties as well (in the sense indicated in the proof below).*

Proof. Indeed, consider any measure $\Lambda \in \mathcal{P}(\Theta)$. Then for any $P_{t+1,T} \in \mathcal{P}_{t+1,T}$ such that $P_{t+1,T|\Theta} = \Lambda$, we will use the formula (3.6) to analyze the implied properties of $\hat{\rho}_t$.

1) Suppose ρ_t is normalized. Then (the symbol $\mathbf{0}$ below denotes a function on Θ that is identically equal to zero)

$$\hat{\rho}_t(\mathbf{0}; \Lambda) = \hat{\rho}_t\left(\{\rho_t(0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}), \theta \in \Theta\}; P_{t+1,T,\Theta}\right) = \rho_t(0, \dots, 0; P_{t+1,T}) = 0.$$

Thus, $\hat{\rho}_t$ is normalized.

2) Suppose ρ_t is normalized and translation invariant and has the support property. Then for any $V \in \mathbb{R}$, we have

$$\rho_t(V, 0, \dots, 0; P_{t+1,T}) = V + \rho_t(0, 0, \dots, 0; P_{t+1,T}) = V.$$

Therefore, for any $U \in \mathcal{Z}^{t,t}$ and any $a \in \mathbb{R}$, by the support, translation invariant, and normalization properties of ρ_t , we have for any $\theta \in \Theta$,

$$\rho_t(U + a, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}) = \rho_t(U(\theta) + a, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}) = U(\theta) + a. \quad (3.7)$$

Therefore,

$$\begin{aligned} \hat{\rho}_t(U + a; \Lambda) &= \hat{\rho}_t\left(\{\rho_t(U + a, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}), \theta \in \Theta\}; P_{t+1,T,\Theta}\right) \\ &= \rho_t(U + a, 0, \dots, 0; P_{t+1,T}) = a + \rho_t(U, 0, \dots, 0; P_{t+1,T}) \\ &= a + \hat{\rho}_t\left(\{\rho_t(U, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}), \theta \in \Theta\}; P_{t+1,T,\Theta}\right) = a + \hat{\rho}_t(U; \Lambda). \end{aligned}$$

Hence, $\hat{\rho}_t$ is translation invariant.

Similarly, for any $U \in \mathcal{Z}^{t,t}$, noting that $\text{supp}_{t,t}(P_{t+1,T}) = \text{supp}(\Lambda)$, we deduce

$$\begin{aligned}\widehat{\rho}_t(U, \Lambda) &= \rho_t(U, 0, \dots, 0; P_{t+1,T}) = \rho_t(\mathbb{1}_{\text{supp}_{t,t}(P_{t+1,T})}U, 0, \dots, 0; P_{t+1,T}) \\ &= \rho_t(\mathbb{1}_{\text{supp}(\Lambda)}U, 0, \dots, 0; P_{t+1,T}) = \widehat{\rho}_t(\mathbb{1}_{\text{supp}(\Lambda)}U, \Lambda).\end{aligned}$$

Thus, $\widehat{\rho}_t$ also has the support property.

3) Suppose ρ_t is normalized, translation invariant, and monotonic. Then, for all $U, W \in \mathcal{Z}^{t,t}$ such that $U \leq W$, employing (3.7) with $a = 0$, we have

$$\begin{aligned}\widehat{\rho}_t(U; \Lambda) &= \widehat{\rho}_t\left(\{\rho_t(U, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}), \theta \in \Theta\}; P_{t+1,T,\Theta}\right) = \rho_t(U, 0, \dots, 0; P_{t+1,T}) \\ &\leq \rho_t(W, 0, \dots, 0; P_{t+1,T}) = \widehat{\rho}_t\left(\{\rho_t(W, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}), \theta \in \Theta\}; P_{t+1,T,\Theta}\right) \\ &= \widehat{\rho}_t(W; \Lambda),\end{aligned}$$

and this proves the monotonicity of $\widehat{\rho}_t$.

4) If ρ is monotonic, normalized, parameter consistent, translation invariant, and with support property, then $\widehat{\rho}_t$ is also normalized, monotonic, normalized, translation invariant, and has the support property, and in view of Remark 3.8, $\widehat{\rho}_t$ is also parameter consistent. \square

3.3 Time Consistency

We now consider the notion of time consistency of risk filters.

Definition 3.11. Let $t \in \{1, \dots, T-1\}$. For any positive measure μ_{t+1} on $\mathcal{X}^{T-t} \times \Theta$ and for any $x \in \mathcal{X}$, we denote by $\mu_{t+1}(\cdot \| x)$ the measure on $\mathcal{X}^{T-t-1} \times \Theta$ given as

$$\mu_{t+1}(A \times B \| x) = \frac{\mu_{t+1}(\{x\} \times A \times B)}{\mu_{t+1}(\{x\} \times \mathcal{X}^{T-t-1} \times \Theta)}.$$

Clearly, $\mu_{t+1}(\cdot \| x)$ is a probability measure.

In particular, taking $\mu_{t+1} = \delta_\theta \otimes P_{t+1,T|\Theta}$,

$$\delta_\theta \otimes P_{t+1,T|\Theta}(A \times B \| x) = \frac{\delta_\theta \otimes P_{t+1,T|\Theta}(\{x\} \times A \times B)}{\delta_\theta \otimes P_{t+1,T|\Theta}(\{x\} \times \mathcal{X}^{T-t-1} \times \Theta)}. \quad (3.8)$$

It follows from (3.1) and (3.8) that

$$\begin{aligned}\delta_\theta \otimes P_{t+1,T|\Theta}(A \times B \| x) &= \frac{P_{t+1,T|\Theta}(\{x\} \times A)\delta_\theta(B)}{P_{t+1,T|\Theta}(\{x\} \times \mathcal{X}^{T-t-1})} = \frac{P_{t+1,T|\Theta}(\{x\} \times A)}{P_{t+1,t+1|\Theta}(\{x\})}\delta_\theta(B) \\ &=: \widetilde{P}_{t+1,T|\Theta}(A \| x)\delta_\theta(B),\end{aligned} \quad (3.9)$$

which, in view of (3.3) gives

$$\delta_\theta \otimes P_{t+1,T|\Theta}^{\pi^{t,h_t}}(A \times B \| x) = \frac{P_{\theta,t+1,T}^{\pi^{t,h_t}}(\{x\} \times A)}{P_{\theta,t+1,t+1}^{\pi^{t,h_t}}(\{x\})}\delta_\theta(B) =: \widetilde{P}_{\theta,t+1,T}^{\pi^{t,h_t}}(A \| x)\delta_\theta(B). \quad (3.10)$$

The next definition is a version of the dynamic conditional time consistency used in [FR18], adapted to the set-up of the present paper.

Definition 3.12. A dynamic risk filter $\rho = \{\rho_t\}_{t=1,\dots,T}$ is *time consistent* if for any $t = 1, \dots, T - 1$, for any $P_{t+1,T}, Q_{t+1,T} \in \mathcal{P}_{t+1,T}$, such that $P_{t+1,t+1|\Theta} = Q_{t+1,t+1|\Theta}$, and for any functions⁴ $Z_{t,s}(\cdot_{s-t}, \cdot_1), W_{t,s}(\cdot_{s-t}, \cdot_1) \in \mathcal{Z}_{t,s}$, $s = t + 1, \dots, T$, the inequalities

$$\begin{aligned} & \rho_{t+1}(Z_{t,t+1}(x_{t+1}, \cdot_1), \dots, Z_{t,T}(x_{t+1}, \cdot_{T-t-1}, \cdot_1); \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \|x_{t+1})) \\ & \leq \rho_{t+1}(W_{t,t+1}(x_{t+1}, \cdot_1), \dots, W_{t,T}(x_{t+1}, \cdot_{T-t-1}, \cdot_1); \delta_\theta \otimes Q_{t+1,T|\Theta}(\cdot \|x_{t+1})), \quad (3.11) \\ & \quad \forall \theta \in \Theta, \quad \forall x_{t+1} \in \mathcal{X}, \end{aligned}$$

imply that for any function $f_t \in \mathcal{Z}_{t,t}$

$$\begin{aligned} & \rho_t(f_t(\cdot_1), Z_{t,t+1}(\cdot_1, \cdot_1), \dots, Z_{t,T}(\cdot_{T-t}, \cdot_1)); \delta_\theta \otimes P_{t+1,T|\Theta}) \\ & \leq \rho_t(f_t(\cdot_1), W_{t,t+1}(\cdot_1, \cdot_1), \dots, W_{t,T}(\cdot_{T-t}, \cdot_1)); \delta_\theta \otimes Q_{t+1,T|\Theta}), \quad \forall \theta \in \Theta. \quad (3.12) \end{aligned}$$

Lemma 3.13. Suppose a dynamic risk filter $\{\rho_t\}_{t=1,\dots,T}$ is normalized, translation invariant, has the support property and is time consistent. Let $\theta \in \Theta$ be fixed. Then the function on $\mathcal{Z}_{t,t+1} \times \mathcal{P}(\mathcal{X}^{T-t} \times \Theta)$ given as

$$\rho_t(0, w, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta})$$

depends only on the probability $P_{t+1,t+1|\theta}$ and on the function $w(\cdot, \theta)$.

Proof. For any $P_{t+1,T}$ and $x \in \mathcal{X}$, the support property of ρ_{t+1} implies that

$$\rho_{t+1}(w(x, \cdot), 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \|x)) = \rho_{t+1}(w(x, \theta), 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \|x)).$$

Then, by the translation invariance and the normalization properties of ρ_{t+1} we obtain

$$\rho_{t+1}(w(x, \cdot), 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \|x)) = w(x, \theta),$$

which does not depend on $P_{t+2,T|\Theta}$. Hence, for any $Q_{t+1,T} \in \mathcal{P}_{t+1,T}$ we have

$$\rho_{t+1}(w(x, \cdot), 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \|x)) = \rho_{t+1}(w(x, \cdot), 0, \dots, 0; \delta_\theta \otimes Q_{t+1,T|\Theta}(\cdot \|x)).$$

If in addition, $P_{t+1,t+1|\Theta} = Q_{t+1,t+1|\Theta}$, then, by the time consistency,

$$\rho_t(0, w, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}) = \rho_t(0, w, 0, \dots, 0; \delta_\theta \otimes Q_{t+1,T|\Theta}).$$

which proves that only the conditional measure $P_{t+1,t+1|\Theta}$ matters in this calculation. The fact that the knowledge of $w(\cdot, \theta)$ is sufficient, follows from the support property. This concludes the proof. \square

In accordance with the above lemma we define the functions

$$\sigma_t : Z_1^{\mathcal{X}} \times \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{R}, \quad t = 1, \dots, T - 1,$$

as

$$\sigma_t(v; P_{t+1,t+1|\theta}) = \rho_t(0, w, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}), \quad (3.13)$$

where $w(\cdot, \theta) \equiv v(\cdot)$ and can be arbitrary otherwise. We refer to these functions as *transition risk mappings*.

Note that if ρ_t is normalized, monotonic, translation invariant and has support property, then so is σ_t .

⁴The notation \cdot_k is a place-holder for k variables.

Theorem 3.14. *A dynamic risk filter $\rho = \{\rho_t\}_{t=1,\dots,T}$ is normalized, monotonic, translation invariant, has the support property, is parameter consistent, and time consistent, if and only if the following conditions are satisfied:*

- 1) *Marginal risk mappings $\widehat{\rho}_t : \mathcal{Z}_{t,t} \times \mathcal{P}(\Theta) \rightarrow \mathbb{R}$, $t \in \mathcal{T}$, exist, which are normalized, monotonic, translation invariant, and have the support property;*
- 2) *Transition risk mappings given in (3.13) are such that*
 - (i) *For all $t = 1, \dots, T-1$, $\sigma_t(\cdot; \cdot)$ is normalized, monotonic, translation invariant, and has the support property;*
 - (ii) *For any $P_{t+1,T} \in \mathcal{P}_{t+1,T}$, $t = 1, \dots, T-1$, and for any functions $Z_{t,s} \in \mathcal{Z}_{t,s}$, $s \in \mathcal{T}_t$, we have that⁵*

$$\rho_t(Z_{t,t}, Z_{t,t+1}, \dots, Z_{t,T}; P_{t+1,T}) = \widehat{\rho}_t \left(\left\{ Z_{t,t}(\theta) + \sigma_t(\rho_{t+1}(Z_{t,t+1}(\diamond, \cdot), \dots, Z_{t,T}(\diamond, \cdot_{T-t-1}, \cdot)); \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel \diamond)); P_{t+1,t+1|\theta}, \theta \in \Theta \right\}; P_{t+1,T,\Theta} \right). \quad (3.14)$$

For any function $Z_{T,T} \in \mathcal{Z}_{T,T}$ and $P_{T+1,T} \in \mathcal{P}(\Theta)$

$$\rho_T(Z_{T,T}; P_{T+1,T}) = \widehat{\rho}_T(Z_{T,T}, P_{T+1,T}). \quad (3.15)$$

Proof. We fix $P_{t+1,T} \in \mathcal{P}_{t+1,T}$ and $Z_{t,s} \in \mathcal{Z}_{t,s}$, $s = t, \dots, T$. Since t is fixed, to alleviate the notations we simply write Z_s , instead of $Z_{t,s}$ in this proof.

Since the risk filter is parameter consistent, Theorem 3.9 yields the existence of mappings $\widehat{\rho}_t$ such that

$$\rho_t(Z_t, Z_{t+1}, \dots, Z_T; P_{t+1,T}) = \widehat{\rho}_t \left(\left\{ \rho_t(Z_t, Z_{t+1}, \dots, Z_T; \delta_\theta \otimes P_{t+1,T|\Theta}), \theta \in \Theta \right\}; P_{t+1,T,\Theta} \right).$$

It follows from Proposition 3.10 that $\widehat{\rho}_t$ is normalized, monotonic, translation invariant, has the support property.

Next, we derive an equivalent expression for the first argument of $\widehat{\rho}_t$ that will prove (3.14). Define the function

$$w(x, \theta) = \rho_{t+1}(Z_{t+1}(x, \cdot), \dots, Z_T(x, \cdot_{T-t-1}, \cdot); \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel x)), \quad x \in \mathcal{X}, \quad \theta \in \Theta.$$

Then, for any fixed $x \in \mathcal{X}$ and $\theta \in \Theta$, we use the support, translation invariance, and the normalization properties in the chain of equations below:

$$\begin{aligned} & \rho_{t+1}(w(x, \cdot), 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel x)) \\ &= \rho_{t+1}(w(x, \theta), 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel x)) \\ &= w(x, \theta) + \rho_{t+1}(0, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel x)) = w(x, \theta) \\ &= \rho_{t+1}(Z_{t+1}(x, \cdot), \dots, Z_T(x, \cdot_{T-t-1}, \cdot); \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel x)). \end{aligned}$$

In view of the assumed time consistency of ρ , the above implies that for every $Z_{t,t} \in \mathcal{Z}_{t,t}$,

$$\rho_t(Z_t, Z_{t+1}, \dots, Z_T; \delta_\theta \otimes P_{t+1,T|\Theta}) = \rho_t(Z_t, w, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}) =: I_1.$$

⁵Recall Definition 3.11. The notation $\sigma_t(\rho_{t+1}(Z_{t,t+1}(\diamond, \theta), \dots, Z_{t,T}(\diamond, \cdot_{T-t-1}, \theta); \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel \diamond)); P_{t+1,t+1|\theta})$, where we use \diamond as place holder, means that $P_{t+1,t+1|\theta}$ acts on $w(x, \theta) = \rho_{t+1}(Z_{t,t+1}(x, \theta), \dots, f_{t,T}(x, \cdot_{T-t-1}, \theta); \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \parallel x))$ as a function of x .

Thus, by using the translation invariance and the support properties again, we conclude that, for all $\theta \in \Theta$,

$$I_1 = \rho_t(Z_t(\theta), w, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}) = Z_t(\theta) + \rho_t(0, w, 0, \dots, 0; \delta_\theta \otimes P_{t+1,T|\Theta}).$$

Using (3.13), we get

$$I_1 = Z_t(\theta) + \sigma_t(w(\cdot, \theta), P_{t+1,t+1|\theta}).$$

Finally, from here, noting that by support property

$$w(x, \theta) = \rho_{t+1}(Z_{t+1}(x, \theta), \dots, Z_T(x, \cdot_{T-t-1}, \theta); \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \|x)), \quad x \in \mathcal{X}, \quad \theta \in \Theta, \quad (3.16)$$

we obtain the representation (3.14). The representation (3.15) follows from the definition of ρ_T and the form of $\hat{\rho}_T$.

Next we prove the converse statement by backward induction in time. For $t = T$, the conditional risk filter (3.15) has all the postulated properties, with the exception of the time consistency, because $\hat{\rho}_T$ does (see Proposition 3.10).

Suppose the conditional risk filters ρ_s , $s = t + 1, \dots, T$ are normalized, monotonic, translation invariant, have the support property, are parameter consistent, and time consistent. We will verify these properties for ρ_t given by formula (3.14). The translation invariance follows from the translation invariance of $\hat{\rho}_t$. The normalization and the monotonicity follow immediately from the normalization and the monotonicity of σ_t , ρ_{t+1} and $\hat{\rho}_t$.

We now verify the support property. For every θ , and x define $\mu_{\theta,x}(\cdot) = \delta_\theta \otimes P_{t+1,T|\Theta}(\cdot \|x)$, $A(\theta) = \text{supp}(P_{t+1,t+1|\theta}) \subset \mathcal{X}$, $B = \text{supp}(P_{t+1,T|\Theta}) \subset \Theta$. Then, by (3.14), (3.16), the support property of $\hat{\rho}_t$ and σ_t , and by Remark 3.3 applied to σ_t , we deduce that

$$\begin{aligned} \rho_t(Z_t, Z_{t+1}, \dots, Z_T; P_{t+1,T}) &= \hat{\rho}_t\left(\left\{Z_t(\theta) + \sigma_t(w(\diamond, \theta); P_{t+1,t+1|\theta}); \theta \in \Theta\right\}; P_{t+1,T,\Theta}\right) \\ &= \hat{\rho}_t\left(\left\{\mathbb{1}_B(\theta)Z_t(\theta) + \mathbb{1}_B(\theta)\sigma_t(w(\diamond, \theta); P_{t+1,t+1|\theta}); \theta \in \Theta\right\}; P_{t+1,T,\Theta}\right) \\ &= \hat{\rho}_t\left(\left\{\mathbb{1}_B(\theta)Z_t(\theta) + \sigma_t(\mathbb{1}_B(\theta)\mathbb{1}_{A(\theta)}(\diamond)w(\diamond, \theta); P_{t+1,t+1|\theta}); \theta \in \Theta\right\}; P_{t+1,T,\Theta}\right). \end{aligned} \quad (3.17)$$

By the assumed support property of ρ_{t+1} , and in view of Remark 3.3 applied to ρ_{t+1} , we obtain, using (3.16) again,

$$\begin{aligned} \mathbb{1}_{A(\theta)}(x)\mathbb{1}_B(\theta)w(x, \theta) &= \rho_{t+1}(\mathbb{1}_{A(\theta)}(x)\mathbb{1}_B(\theta)Z_{t+1}(x, \theta), \mathbb{1}_{A(\theta)}(x)\mathbb{1}_B(\theta)\mathbb{1}_{\text{supp}_{t+2}(\mu_{\theta,x})}Z_{t+2}(x, \cdot, \theta), \\ &\quad \dots, \mathbb{1}_{A(\theta)}(x)\mathbb{1}_B(\theta)\mathbb{1}_{\text{supp}_T(\mu_{\theta,x})}Z_T(x, \cdot_{T-t-1}, \theta); \mu_{\theta,x}), \end{aligned}$$

for every $x \in \mathcal{X}$ and $\theta \in \Theta$. From here and (3.17), combined with the normalization property of ρ_{t+1} , and the fact that $\mathbb{1}_{A(\theta)}(x)\mathbb{1}_B(\theta)\mathbb{1}_{\text{supp}_s(\mu_{\theta,x})} \leq \mathbb{1}_{\text{supp}_s(P_{t+1,T})}$, $s = t, \dots, T$, we obtain the support property of ρ_t .

Next we prove the parameter consistency. Assume that (3.4) is satisfied for a fixed $\bar{\theta} \in \Theta$, and denote by $\bar{P}_{t+1,T} = \delta_{\bar{\theta}} \otimes P_{t+1,T|\Theta}$ and $\bar{Q}_{t+1,T} = \delta_{\bar{\theta}} \otimes Q_{t+1,T|\Theta}$. We note that⁶

$$\bar{P}_{t+1,t+1|\theta} = P_{t+1,t+1|\bar{\theta}} \cdot \mathbb{1}_{\bar{\theta}}(\theta), \quad \bar{P}_{t+1,T,\Theta} = \delta_{\bar{\theta}}, \quad \bar{P}_{t+1,T|\theta}(\cdot \|x) = P_{t+1,T|\bar{\theta}}(\cdot \|x)\mathbb{1}_{\bar{\theta}}(\theta).$$

⁶We use the convention that $\frac{0}{0} = 0$ when considering $\bar{P}_{t+1,t+1|\theta}$ and $\bar{P}_{t+1,T|\theta}(\cdot \|x)$.

Using this, and in view of (3.14), we can write (3.4) as follows (with measures $\bar{P}_{t+1,T}$ and $\bar{Q}_{t+1,T}$ in place of $P_{t+1,T}$ and $Q_{t+1,T}$):

$$\begin{aligned} \hat{\rho}_t & \left(\left\{ Z_t(\theta) + \sigma_t(\rho_{t+1}(Z_{t+1}(\diamond, \theta), \dots, Z_T(\diamond, \cdot_{T-t-1}, \theta); \right. \right. \\ & \quad \left. \left. \delta_{\bar{\theta}}(\cdot) P_{t+1,T|\bar{\theta}}(\cdot \|\diamond) \mathbb{1}_{\bar{\theta}}(\theta); P_{t+1,t+1|\bar{\theta}} \mathbb{1}_{\bar{\theta}}(\theta)), \theta \in \Theta \right\}; \delta_{\bar{\theta}} \right) \\ & \leq \hat{\rho}_t \left(\left\{ W_t(\theta) + \sigma_t(\rho_{t+1}(W_{t+1}(\diamond, \theta), \dots, W_T(\diamond, \cdot_{T-t-1}, \theta); \right. \right. \\ & \quad \left. \left. \delta_{\bar{\theta}}(\cdot) Q_{t+1,T|\bar{\theta}}(\cdot \|\diamond) \mathbb{1}_{\bar{\theta}}(\theta); Q_{t+1,t+1|\bar{\theta}} \mathbb{1}_{\bar{\theta}}(\theta)), \theta \in \Theta \right\}; \delta_{\bar{\theta}} \right). \end{aligned}$$

By the support property of $\hat{\rho}_t$ and σ_t , and by the normalization and monotonicity of $\hat{\rho}_t$, we obtain that

$$\begin{aligned} & Z_t(\bar{\theta}) + \sigma_t(\rho_{t+1}(Z_{t+1}(\diamond, \bar{\theta}), \dots, Z_T(\diamond, \cdot_{T-t-1}, \bar{\theta}); \delta_{\bar{\theta}}(\cdot) P_{t+1,T|\bar{\theta}}(\cdot \|\diamond); P_{t+1,t+1|\bar{\theta}}) \\ & \leq W_t(\bar{\theta}) + \sigma_t(\rho_{t+1}(W_{t+1}(\diamond, \bar{\theta}), \dots, W_T(\diamond, \cdot_{T-t-1}, \bar{\theta}); \delta_{\bar{\theta}}(\cdot) Q_{t+1,T|\bar{\theta}}(\cdot \|\diamond); Q_{t+1,t+1|\bar{\theta}})), \end{aligned}$$

for any $\bar{\theta} \in \Theta$. From here, applying $\hat{\rho}_t$ to both sides, since we assumed that $P_{t+1,T,\Theta} = Q_{t+1,T,\Theta}$, employing monotonicity of $\hat{\rho}_t$, we obtain (3.5), and the parameter consistency of ρ_t is proved.

Finally, let us verify the time consistency at time t . If the inequalities (3.11) are satisfied, then it follows from the monotonicity of σ_t with respect to its first argument that for all $\theta \in \Theta$

$$\begin{aligned} G_1(\theta) & := \sigma_t(\rho_{t+1}(Z_{t+1}(\diamond, \theta), \dots, Z_T(\diamond, \cdot_{T-t-1}, \theta); \delta_{\theta} \otimes P_{t+1,T|\Theta}(\cdot \|\diamond)); P_{t+1,t+1|\theta}) \\ & \leq \sigma_t(\rho_{t+1}(W_{t+1}(\diamond, \theta), \dots, W_T(\diamond, \cdot_{T-t-1}, \theta); \delta_{\theta} \otimes Q_{t+1,T|\Theta}(\cdot \|\diamond)); P_{t+1,t+1|\theta}) =: G_2(\theta). \end{aligned}$$

Then, from the monotonicity of $\hat{\rho}_t$ we get, for any function $f_t \in \mathcal{Z}_{t,t}$,

$$\hat{\rho}_t(f_t + G_1; \delta_{\theta}) \leq \hat{\rho}_t(f_t + G_2; \delta_{\theta}), \quad \theta \in \Theta.$$

From here, using the support property of $\hat{\rho}_t$, σ_t and ρ_{t+1} , along (3.14), we obtain (3.12), and thus time consistency at t is verified.

By induction, all properties hold true for $t = 1, \dots, T$, and the proof is complete. \square

Example 3.15. We consider a very special conditional risk filter, given as the expectation of an additive functional under the measure $P_{t+1,T}$. Specifically, we let

$$\begin{aligned} \rho_t(Z_{t,t}, \dots, Z_{t,T}; P_{t+1,T}) & = \int_{\mathcal{X}^{T-t} \times \Theta} \sum_{k=t}^T Z_{t,k}(x_{t+1}, \dots, x_k, \theta) P_{t+1,T}(dx_{t+1}, \dots, dx_T, d\theta) \\ & = E_{P_{t+1,T}} \sum_{k=t}^T Z_{t,k}. \end{aligned}$$

Clearly, this ρ_t is normalized, monotonic, translation invariant, and has the support property (cf. Definition 3.2).

Next, note that for this ρ_t the inequality (3.4) becomes (cf. (3.1))

$$\int_{\mathcal{X}^{T-t}} \sum_{k=t}^T Z_{t,k}(x_{t+1:k}, \theta) P_{t+1,T|\theta}(dx_{t+1}, \dots, dx_T) \leq \int_{\mathcal{X}^{T-t}} \sum_{k=t}^T Z_{t,k}(x_{t+1:k}, \theta) Q_{t+1,T|\theta}(dx_{t+1}, \dots, dx_T),$$

for any $\theta \in \Theta$. Assuming that $P_{t+1,T,\Theta} = Q_{t+1,T,\Theta}$, multiplying the last inequality by $P_{t+1,T,\Theta}(\theta)$, and summing up with respect to $\theta \in \Theta$, the inequality (3.5) follows, and hence the parameter consistency is true.

The time consistency follows by similar arguments. Indeed, (3.11) becomes (cf. (3.9))

$$\begin{aligned} & \int_{\mathcal{X}^{T-t-1}} \sum_{k=t+1}^T Z_{t,k}(x_{t+1}, x_{t+2:k}, \theta) \tilde{P}_{t+1,T|\theta}(dx_{t+2}, \dots, dx_T \| x_{t+1}) \\ & \leq \int_{\mathcal{X}^{T-t-1}} \sum_{k=t+1}^T Z_{t,k}(x_{t+1}, x_{t+2:k}, \theta) \tilde{Q}_{t+1,T|\theta}(dx_{t+2}, \dots, dx_T \| x_{t+1}), \end{aligned}$$

for any $x_{t+1} \in \mathcal{X}$ and $\theta \in \Theta$. Assuming that $P_{t+1,t+1|\Theta} = Q_{t+1,t+1|\Theta}$, multiplying both parts by $P_{t+1,t+1|\theta}(x_{t+1})$, and noting that (cf. (3.9))

$$\tilde{P}_{t+1,T|\theta}(\cdot \| x_{t+1}) P_{t+1,t+1|\theta}(x_{t+1}) = P_{t+1,T|\theta}(x_{t+1}, \cdot),$$

for any function $f_t \in \mathcal{Z}_{t,t}$ we have (cf. (3.9))

$$\begin{aligned} f_t(\theta) + \int_{\mathcal{X}^{T-t-1}} \sum_{k=t+1}^T Z_{t,k}(x_{t+1}, x_{t+2:k}, \theta) P_{t+1,T|\theta}(\{x_{t+1}\}, dx_{t+2}, \dots, dx_T) \\ \leq f_t(\theta) + \int_{\mathcal{X}^{T-t-1}} \sum_{k=t+1}^T Z_{t,k}(x_{t+1}, x_{t+2:k}, \theta) Q_{t+1,T|\theta}(\{x_{t+1}\}, dx_{t+2}, \dots, dx_T), \end{aligned}$$

After summing up with respect to x_{t+1} we obtain (3.12), and thus the time consistency is proved.

We complete this example by observing that in the this case we have that $\hat{\rho}_t(f, P') = E_{P'}(f)$, for $f \in \mathcal{Z}_{t,t}$ and for $P' \in \mathcal{P}(\Theta)$, and that $\sigma_t(v; P'') = E_{P''}(v)$, for $v \in \mathcal{Z}_1^{\mathcal{X}}$ and $P'' \in \mathcal{P}(\mathcal{X})$.

Example 3.16. Let us cast Example 3.15 in the setup of Section 2. For this, we fix a history $h_t = (x_1, \dots, x_t) \in \mathcal{H}_t$ and $\pi \in \Pi$, and we take

$$Z_{t,t}(\theta) := Z_{\theta,t,t}^{\pi, h_t} = c_t(x_t, \pi_t(h_t), \theta),$$

$$Z_{t,s}(x_{t+1}, \dots, x_s, \theta) := Z_{\theta,t,s}^{\pi, h_t, x_{t+1}, \dots, x_s} = c_s(x_s, \pi_s^{t, h_t}(x_{t+1}, \dots, x_s), \theta), s = t+1, \dots, T,$$

and

$$P_{t+1,T} = P_{t+1,T}^{\pi_t, h_t}.$$

The conditional risk filter of Example 3.15 becomes a conditional expectation (cf. Lemma 2.2)

$$\begin{aligned} & \rho_t \left(c_t(x_t, \pi_t(h_t), \cdot), c_{t+1}(\cdot, \pi_{t+1}(h_t, \cdot), \cdot), \dots, c_T(\cdot, \pi_T(h_t, \cdot, \dots, \cdot), \cdot), P_{t+1,T}^{\pi_t, h_t} \right) \\ & = E^\pi \left[c_t(x_t, \pi_t(h_t), \Theta) + \sum_{s=t+1}^T c_s(\hat{X}_s, \pi_s^{t, h_t}(\hat{X}_{t+1}, \dots, \hat{X}_s), \Theta) \mid \hat{H}_t = h_t \right], \end{aligned} \quad (3.18)$$

for $t = 1, \dots, T$, where we use the standard convention that an empty sum is zero (i.e. $\sum_{s=T+1}^T \dots = 0$ in our case).

In view of (2.7) we also have

$$\begin{aligned}
& \rho_t \left(c_t(x_t, \pi_t(h_t), \cdot), c_{t+1}(\cdot, \pi_{t+1}(h_t, \cdot), \cdot), \dots, c_T(\cdot, \pi_T(h_t, \cdot, \dots, \cdot), \cdot), P_{t+1, T}^{\pi_t, h_t} \right) \\
&= \widehat{\rho}_t \left(\left\{ c_t(x_t, \pi_t(h_t), \theta) + \sigma_t \left(c_{t+1}(\diamond, \pi_{t+1}(h_t, \diamond), \cdot), c_{t+2}(\cdot, \pi_{t+2}(h_t, \diamond, \cdot), \cdot), \right. \right. \right. \\
&\quad \left. \left. \left. \dots, c_T(\cdot, \pi_T(h_t, \diamond, \cdot, \dots, \cdot), \cdot); \delta_\theta \otimes P_{t+1, T | \Theta}^{\pi_t, h_t}(\cdot \| \diamond); P_{t+1, t+1 | \theta}^{\pi_t, h_t}, \theta \in \Theta \right\}; \xi_t^{\pi, h_t} \right) \\
&= \widehat{\rho}_t \left(\left\{ c_t(x_t, \pi_t(h_t), \theta) + \sigma_t \left(c_{t+1}(\diamond, \pi_{t+1}(h_t, \diamond), \cdot), c_{t+2}(\cdot, \pi_{t+2}(h_t, \diamond, \cdot), \cdot), \right. \right. \right. \\
&\quad \left. \left. \left. \dots, c_T(\cdot, \pi_T(h_t, \diamond, \cdot, \dots, \cdot), \cdot); P_{\theta, t+1, T}^{\pi_t, h_t}(\{\diamond\} \times \cdot) \delta_\theta(\cdot); P_{\theta, t+1}^{\pi_t, h_t} \right), \theta \in \Theta \right\}; \xi_t^{\pi, h_t} \right),
\end{aligned}$$

where in the last equality we used (3.2) and (3.10), and where, for a function f on Θ and a measure ξ on Θ ,

$$\widehat{\rho}_t \left(\{f(\theta), \theta \in \Theta\}; \xi_t^{\pi, h_t} \right) = \widehat{\rho}_t(f; \xi_t^{\pi, h_t}) = \int_{\Theta} f(\theta) \xi_t^{\pi, h_t}(d\theta) = E_{\xi_t^{\pi, h_t}}(f), \quad (3.19)$$

and where, for a function v on \mathcal{X} , we have (cf. (3.2), (3.13), and (3.3))

$$\sigma_t \left(v, P_{\theta, t+1}^{\pi_t, h_t} \right) = \int_{\mathcal{X}} v(x) P_{\theta, t+1}^{\pi_t, h_t} d(x) = E_{P_{\theta, t+1}^{\pi_t, h_t}}(v). \quad (3.20)$$

Example 3.17. In the previous example we proceeded from ρ to σ (via $\widehat{\rho}$). Here, we will do the opposite.

In clinical trials, the potency of a drug is characterized by an unknown parameter θ . The purpose of the trials is to estimate θ and to determine the optimal dose. Let us assume for simplicity that θ is the optimal dose. If a dose u_1 is administered to a patient, a response X_2 is observed (the subscript 2 indicates that X_2 is not known when u_1 is determined). X_2 is a Bernoulli random variable, with $X_2 = 1$ representing toxic response, and $X_2 = 0$ nontoxic. The probability of toxic response is a function of θ and u_1 , that is, $P[X_2 = 1] = \Psi(\theta, u_1)$. The ‘‘cost’’ is $c(\theta, u_1)$; it depends on both the applied and best doses. The cost is not observed; we only know whether the patient was toxic or not. In the second stage, the dose u_2 is administered to the next patient, the patient’s response X_3 observed, and cost $c(\theta, u_2)$ incurred. The process continues for T stages, with u_T being the final dose recommendation, whose cost is equal to $c(\theta, u_T)$. For example, the cost may have the form $c(\theta, u) = |u - \theta|$ to penalize for the over- and under-dosage. It is never observed.

The problem can be cast to our setting. The state space \mathcal{X} is $\{0, 1\}$, while the unknown parameter space Θ is an interval of the real line or a finite subset of the real line. Given the set-up adopted in this paper, we assume that Θ is a finite subset of the real line. The transition kernel does not depend on X at all; the distribution of the next X_{t+1} depends on θ and u :

$$K_\theta(0|x, u) = 1 - \Psi(\theta, u), \quad K_\theta(1|x, u) = \Psi(\theta, u).$$

Thus, we have

$$P_{\theta, t+1}^{\pi_t, h_t}(y) = K_\theta(y|x_t, \pi_t(h_t)) = \mathbb{1}_{y=0}(1 - \Psi(\theta, \pi_t(h_t))) + \mathbb{1}_{y=1}\Psi(\theta, \pi_t(h_t)), \quad y \in \{0, 1\}.$$

There is a considerable leverage in choosing the form of σ_t in a way consistent with the above set-up. For example, one can choose σ_t in terms of the entropic risk measures, as follows

$$\sigma(w, P) = \frac{1}{\kappa} \ln \int_{\mathcal{X}} e^{\kappa w(y)} P(dy),$$

for a function f on \mathcal{X} , $P \in \mathcal{P}(\mathcal{X})$ and a constant $\kappa > 0$. Consequently, for $t = 1, \dots, T-1$, using (2.5) and (2.9) we obtain

$$\begin{aligned}\sigma_t(w(\cdot, \theta); P_{t+1, t+1|\theta}^{\pi^{t, h_t}}) &= \frac{1}{\varkappa} \ln \left((1 - \Psi(\theta, \pi_t(h_t))) e^{\varkappa w(0, \theta)} + \Psi(\theta, \pi_t(h_t)) e^{\varkappa w(1, \theta)} \right) \\ &= \frac{1}{\varkappa} \ln \int_{\mathcal{X}} e^{\varkappa w(y, \theta)} P_{\theta, t+1}^{\pi^{t, h_t}}(dy),\end{aligned}$$

with $\sigma_T = 0$.

Now, for a function f on Θ and a measure $\xi \in \mathcal{P}(\Theta)$, let

$$\widehat{\rho}_t(\{f(\theta), \theta \in \Theta\}; \xi) = \widehat{\rho}_t(f; \xi) = \frac{1}{\varkappa} \ln \int_{\Theta} e^{\varkappa f(\theta)} \xi(d\theta), \quad t \in \mathcal{T}.$$

Given the above, we obtain for $t = T$

$$\widehat{\rho}_T(\{v(\theta), \theta \in \Theta\}; \xi_T^{\pi, h_T}) = \frac{1}{\varkappa} \ln \int_{\Theta} e^{\varkappa v(\theta)} \xi_T^{\pi, h_T}(d\theta),$$

and for $t = 1, \dots, T-1$,

$$\begin{aligned}\widehat{\rho}_t(\{v(\theta) + \sigma_t(w(\cdot, \theta); P_{t+1, t+1|\theta}^{\pi^{t, h_t}}), \theta \in \Theta\}; \xi_t^{\pi, h_t}) &= \frac{1}{\varkappa} \ln \int_{\Theta} \int_{\mathcal{X}} e^{\varkappa(v(\theta) + w(y, \theta))} P_{\theta, t+1}^{\pi^{t, h_t}}(dy) \xi_t^{\pi, h_t}(d\theta) \\ &= \frac{1}{\varkappa} \ln E^\pi [e^{\varkappa(v(\Theta) + w(X_{t+1}, \Theta))} | \widehat{H}_t = h_t],\end{aligned}$$

where the last equality follows from Lemma 2.4.

We will now derive a generic formula for ρ_t , generated by (3.14) and σ_t and $\widehat{\rho}_t$ as above, in case of the generic cost functions as in (2.15) and (2.16). Let us fix an admissible strategy π . For $t = T$ we have

$$\begin{aligned}\rho_T(c_T(x_T, \pi_T(h_T), \cdot), P_{T+1, T}^{\pi^{T, h_T}}) &= \widehat{\rho}_T(\{c_T(x_T, \pi_T(h_T), \theta), \theta \in \Theta\}; P_{T+1, T, \Theta}^{\pi^{T, h_T}}) \\ &= \widehat{\rho}_T(c_T(x_T, \pi_T(h_T), \cdot); P_{T+1, T, \Theta}^{\pi^{T, h_T}}) = \widehat{\rho}_T(c_T(x_T, \pi_T(h_T), \cdot); \xi_T^{\pi, h_T}) \\ &= \frac{1}{\varkappa} \ln \int_{\Theta} e^{\varkappa c_T(x_T, \pi_T(h_T), \theta)} \xi_T^{\pi, h_T}(d\theta) \\ &= \frac{1}{\varkappa} \ln E^\pi (e^{\varkappa c_T(x_T, \pi_T(h_T), \Theta)} | \widehat{H}_T = h_T).\end{aligned}$$

Now, note that

$$\rho_T(c_T(x_T, \pi_T(h_T), \cdot), \delta_\theta) = c_T(x_T, \pi_T(h_T), \theta),$$

and thus

$$\begin{aligned}\sigma_{T-1}(\rho_T(c_T(\diamond, \pi_T(h_{T-1}, \diamond), \theta); \delta_\theta); P_{T, T|\theta}^{\pi^{T-1, h_{T-1}}}) &= \sigma_{T-1}(c_T(\diamond, \pi_T(h_{T-1}, \diamond), \theta); P_{T, T|\theta}^{\pi^{T-1, h_{T-1}}}) \\ &= \frac{1}{\varkappa} \ln \int_{\mathcal{X}} e^{\varkappa c_T(x_T, \pi_T(h_{T-1}, x_T), \theta)} P_{\theta, T}^{\pi^{T-1, h_{T-1}}}(dx_T).\end{aligned}$$

So, for $t = T-1$, we have

$$\begin{aligned}\rho_{T-1}(c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-1}), \cdot), c_T(\cdot, \pi_T(h_{T-1}, \cdot), \cdot), P_{T, T}^{\pi^{T-1, h_{T-1}}}) \\ = \widehat{\rho}_{T-1}(\{c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-1}), \theta) + \sigma_{T-1}(\rho_T(c_T(\diamond, \pi_T(h_{T-1}, \diamond), \cdot); \delta_\theta); P_{T, T|\theta}^{\pi^{T-1, h_{T-1}}}) \\ \theta \in \Theta\}; P_{T, T, \Theta}^{\pi^{T-1, h_{T-1}}})\end{aligned}$$

$$\begin{aligned}
&= \widehat{\rho}_{T-1} \left(\left\{ c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-1}), \theta) + \sigma_{T-1}(\rho_T(c_T(\diamond, \pi_T(h_{T-1}, \diamond), \cdot); \delta_\theta); P_{T,T|\theta}^{\pi^{T-1}, h_{T-1}})) \right. \right. \\
&\quad \left. \left. \theta \in \Theta \right\}; \xi_{T-1}^{\pi, h_{T-1}} \right) \\
&= \frac{1}{\varkappa} \ln \int_{\Theta} \int_{\mathcal{X}} e^{\varkappa(c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-1}), \theta) + c_T(x_T, \pi_T(h_{T-1}, x_T), \theta))} P_{\theta, T}^{\pi^{T-1}, h_{T-1}}(dx_T) \xi_{T-1}^{\pi, h_{T-1}}(d\theta) \\
&= \frac{1}{\varkappa} \ln E^\pi \left(e^{\varkappa(c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-1}), \Theta) + c_T(X_T, \pi_T(h_{T-1}, X_T), \Theta))} | \widehat{H}_{T-1} = h_{T-1} \right),
\end{aligned}$$

where we used (2.5) and (2.9) for the second to the last equality, and where the last equality follows from Lemma 2.4.

Next, note that

$$\begin{aligned}
&\rho_{T-1}(c_{T-1}(\diamond, \pi_{T-1}(h_{T-2}, \diamond), \cdot), c_T(\cdot, \pi_T(h_{T-2}, \diamond, \cdot), \cdot); \delta_\theta \otimes P_{T,T|\Theta}^{\pi^{T-1}, (h_{T-2}, \diamond)}(\cdot \| \diamond)) \\
&= \frac{1}{\varkappa} \ln \int_{\mathcal{X}} e^{\varkappa(c_{T-1}(\diamond, \pi_{T-1}(h_{T-2}, \diamond), \theta) + c_T(y, \pi_T(h_{T-2}, \diamond, y), \theta))} P_{\theta, T}^{\pi^{T-1}, (h_{T-2}, \diamond)}(dy),
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_{T-2}(\rho_{T-1}(c_{T-1}(\diamond, \pi_{T-1}(h_{T-2}, \diamond), \theta), c_T(\cdot, \pi_T(h_{T-2}, \diamond, \cdot), \theta); \\
&\quad \delta_\theta \otimes P_{T,T|\Theta}^{\pi^{T-1}, (h_{T-2}, \diamond)}(\cdot \| \diamond)); P_{T-1, T-1|\theta}^{\pi^{T-2}, h_{T-2}}) \\
&= \frac{1}{\varkappa} \ln \int_{\mathcal{X}} e^{\varkappa(c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-2}, x_{T-1}), \theta) + c_T(x_T, \pi_T(h_{T-2}, x_{T-1}, x_T), \theta))} \\
&\quad P_{\theta, T}^{\pi^{T-1}, (h_{T-2}, x_{T-1})}(dx_T) P_{\theta, T-1}^{\pi^{T-2}, h_{T-2}}(dx_{T-1}).
\end{aligned}$$

We take now $t = T - 2$. In this case,

$$\begin{aligned}
&\rho_{T-2, T}(c_{T-2}(x_{T-2}, \pi_{T-2}(h_{T-2}), \cdot), c_{T-1}(\diamond, \pi_{T-1}(h_{T-2}, \diamond), \cdot), c_T(\cdot, \pi_T(h_{T-2}, \diamond, \cdot), \cdot), P_{T-1, T}^{\pi^{T-2}, h_{T-2}}) \\
&= \widehat{\rho}_{T-2} \left(\left\{ c_{T-2}(x_{T-2}, \pi_{T-2}(h_{T-2}), \theta) + \sigma_{T-2}(\rho_{T-1}(c_{T-1}(\diamond, \pi_{T-1}(h_{T-2}, \diamond), \theta), c_T(\cdot, \pi_T(h_{T-2}, \diamond, \cdot), \theta); \right. \right. \\
&\quad \left. \left. \delta_\theta \otimes P_{T,T|\Theta}^{\pi^{T-1}, (h_{T-2}, \diamond)}(\cdot \| \diamond); P_{T-1, T-1|\theta}^{\pi^{T-2}, h_{T-2}}), \theta \in \Theta \right\}; P_{T-1, T|\Theta}^{\pi^{T-2}, h_{T-2}} \right) \\
&= \widehat{\rho}_{T-2} \left(\left\{ c_{T-2}(x_{T-2}, \pi_{T-2}(h_{T-2}), \theta) + \sigma_{T-2}(\rho_{T-1}(c_{T-1}(\diamond, \pi_{T-1}(h_{T-2}, \diamond), \theta), c_T(\cdot, \pi_T(h_{T-2}, \diamond, \cdot), \theta); \right. \right. \\
&\quad \left. \left. \delta_\theta \otimes P_{T,T|\Theta}^{\pi^{T-1}, (h_{T-2}, \diamond)}(\cdot \| \diamond); P_{T-1, T-1|\theta}^{\pi^{T-2}, h_{T-2}}), \theta \in \Theta \right\}; \xi_{T-2}^{\pi, h_{T-2}} \right) \\
&= \frac{1}{\varkappa} \ln \int_{\Theta} \int_{\mathcal{X}} \int_{\mathcal{X}} e^{\varkappa(c_{T-2}(x_{T-2}, \pi_{T-2}(h_{T-2}), \theta) + c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-2}, x_{T-1}), \theta) + c_T(x_T, \pi_T(h_{T-2}, x_{T-1}, x_T), \theta))} \\
&\quad P_{\theta, T}^{\pi^{T-1}, h_{T-2}, x_{T-1}}(dx_T) P_{\theta, T-1}^{\pi^{T-2}, h_{T-2}}(dx_{T-1}) \xi_{T-2}^{\pi, h_{T-2}}(d\theta) \\
&= \frac{1}{\varkappa} \ln E^\pi \left(e^{\varkappa(c_{T-2}(x_{T-2}, \pi_{T-2}(h_{T-2}), \Theta) + c_{T-1}(X_{T-1}, \pi_{T-1}(h_{T-2}, X_{T-1}), \Theta) + c_T(X_T, \pi_T(h_{T-2}, X_{T-1}, X_T), \Theta))} \right. \\
&\quad \left. | \widehat{H}_{T-2} = h_{T-2} \right),
\end{aligned}$$

where the last equality follows from Lemma 2.4.

Proceeding in the analogous way for $t = T - 3, \dots, 1$ we finally obtain

$$\begin{aligned}
&\rho_{1, T}(c_1(x_1, \pi_1(h_1), \cdot), \dots, c_T(\cdot, \pi_T(h_1, \cdot, \dots, \cdot), \cdot), P_{2, T}^{\pi^1, h_1}) \\
&= \frac{1}{\varkappa} \ln E^\pi \left(e^{\varkappa \sum_{k=1}^T c_k(X_k, \pi_k(h_k), \Theta)} | \widehat{H}_1 = h_1 \right),
\end{aligned}$$

which gives us the risk-sensitive criterion with entropic utility (cf. [BR14, DL14]).

4 Recursive Risk Filters

Let us fix $t \in \{1, \dots, T-1\}$. To alleviate notation, for all $\pi \in \Pi$, we write for fixed functions $Z_{t,s}(\cdot, s-t, \cdot) \in \mathcal{Z}_{t,s}$, $s = t, \dots, T$. Since t is fixed, we will again simply write Z_s instead of $Z_{t,s}$, for $s \in \mathcal{T}_t$.

$$\begin{aligned} v_t^\pi(h_t) &= \rho_t(Z_t, Z_{t+1}, \dots, Z_T; P_{t+1,T}^{\pi^t, h_t}) \\ \tilde{v}_{t+1}^{\pi, \theta}((h_t, x_{t+1})) &:= \rho_{t+1}(Z_{t+1}(x_{t+1}, \cdot), \dots, Z_T(x_{t+1}, \cdot, T-t-1, \cdot); \delta_\theta \otimes P_{t+1,T|\Theta}^{\pi^t, h_t}(\cdot \| x_{t+1})) \\ &= \rho_{t+1}(Z_{t+1}(x_{t+1}, \cdot), \dots, Z_T(x_{t+1}, \cdot, T-t-1, \cdot); P_{\theta, t+1, T}^{\pi^t, h_t}(\{x_{t+1}\} \times \cdot) \delta_\theta(\cdot)), \end{aligned} \quad (4.1)$$

where for the last equality we used (3.10).

The quantity $v_t^\pi(h_t)$ evaluates the policy π at the time t and with the history h_t in the original problem.

Recall that (cf. (3.2)) $P_{t+1,T,\Theta}^{\pi^t, h_t} = \xi_t^{\pi, h_t}$. Thus, the key equation (3.14) can be written more compactly as follows:

$$\begin{aligned} v_t^\pi(h_t) &= \hat{\rho}_t \left(\left\{ Z_t(\theta) + \sigma_t(\rho_{t+1}(Z_{t+1}(\diamond, \cdot), \dots, Z_T(\diamond, \cdot, T-t-1, \cdot); \right. \right. \\ &\quad \left. \left. \delta_\theta \otimes P_{t+1,T|\Theta}^{\pi^t, h_t}(\cdot \| \diamond)); P_{t+1,t+1|\theta}^{\pi^t, h_t}, \theta \in \Theta \right\}; P_{t+1,T,\Theta}^{\pi^t, h_t} \right) \\ &= \hat{\rho}_t \left(\left\{ Z_t(\theta) + \sigma_t(Z_{t+1}(\diamond, \cdot), \dots, Z_T(\diamond, \cdot, T-t-1, \cdot); \right. \right. \\ &\quad \left. \left. \delta_\theta \otimes P_{t+1,T|\Theta}^{\pi^t, h_t}(\cdot \| \diamond)); P_{t+1,t+1|\theta}^{\pi^t, h_t}, \theta \in \Theta \right\}; \xi_t^{\pi, h_t} \right) \\ &= \hat{\rho}_t \left(\left\{ Z_t(\theta) + \sigma_t(\tilde{v}_{t+1}^{\pi, \theta}((h_t, \diamond)); P_{t+1,t+1|\theta}^{\pi^t, h_t}), \theta \in \Theta \right\}; \xi_t^{\pi, h_t} \right), \\ &= \hat{\rho}_t \left(\left\{ Z_t(\theta) + \sigma_t(\tilde{v}_{t+1}^{\pi, \theta}((h_t, \diamond)); P_{\theta, t+1}^{\pi^t, h_t}), \theta \in \Theta \right\}; \xi_t^{\pi, h_t} \right), \end{aligned} \quad (4.2)$$

with σ_t given in (3.13), and where we used (2.12) in the last equality.

Note that in equation (4.2) we have $\tilde{v}_{t+1}^{\pi, \theta}$ on the right hand side. Thus, this equation does not provide a convenient recursion for the quantities v_t^π . This leads us to the following concept,

Definition 4.1. A dynamic risk filter ρ is called *recursive* if it satisfies the properties stated in Theorem 3.14 and

$$v_t^\pi(h_t) = \hat{\rho}_t \left(\left\{ Z_{t,t}(\theta) + \sigma_t(v_{t+1}^\pi((h_t, \diamond)); P_{\theta, t+1}^{\pi^t, h_t}), \theta \in \Theta \right\}; \xi_t^{\pi, h_t} \right),$$

for $t = T-1, \dots, 1$, with

$$v_T^\pi(h_T) = \hat{\rho}_T \left(\left\{ Z_{T,T}(\theta), \theta \in \Theta \right\}; \xi_T^{\pi, h_T} \right).$$

4.1 Examples of recursive dynamic risk-filters

We will show here that risk-filters considered in Example 3.16 and Example 3.17 are recursive.

In the case of the additive risk rewards, that is Example 3.16, using (3.18) we get

$$\begin{aligned} v_t^\pi(h_t) &= \rho_t \left(c_t(x_t, \pi_t(h_t), \cdot), c_{t+1}(\cdot, \pi_{t+1}(h_t, \cdot), \cdot), \right. \\ &\quad \left. c_{t+2}(\cdot, \pi_{t+2}(h_t, \cdot, \cdot), \cdot), \dots, c_T(\cdot, \pi_T(h_t, \cdot, \dots, \cdot), \cdot), P_{t+1,T}^{\pi^t, h_t} \right) \\ &= E^\pi \left[c_t(x_t, \pi_t(h_t), \Theta) + \sum_{s=t+1}^T c_s(\hat{X}_s, \pi_s(\hat{X}_{t+1}, \dots, \hat{X}_s), \Theta) \mid \hat{H}_t = h_t \right], \end{aligned} \quad (4.3)$$

for $t \in \mathcal{T}$, where again we use the standard convention that an empty sum is zero (i.e. $\sum_{s=t+T}^T \dots = 0$ in our case).

Given the above, we obtain that the risk filter considered in Example 3.16 is recursive:

Lemma 4.2. *Let $\widehat{\rho}_t$ and σ_t be given as in (3.19) and (3.20), respectively. We have*

$$v_t^\pi(h_t) = \widehat{\rho}_t \left(\left\{ c_t(x_t, \pi_t(h_t), \theta) + \sigma_t(v_{t+1}^\pi((h_t, \cdot)); P_{\theta, t+1}^{\pi^t, h_t}), \theta \in \Theta \right\}; \xi_t^{\pi, h_t} \right) \quad (4.4)$$

for $t = T-1, \dots, 1$, with

$$v_T^\pi(h_T) = \int_{\Theta} c_T(x_T, \pi_T(h_T), \theta) \xi_T^{\pi, h_T}(d\theta). \quad (4.5)$$

Proof. Fix $t \in \{1, \dots, T-1\}$. First, using (4.3) and the tower property of conditional expectations we get

$$\begin{aligned} v_t^\pi(\widehat{H}_t) &= E^\pi \left[c_t(\widehat{X}_t, \pi_t(\widehat{H}_t), \Theta) + \sum_{s=t+1}^T c_s(\widehat{X}_s, \pi_s^{t, h_t}(\widehat{X}_{t+1}, \dots, \widehat{X}_s), \Theta) \mid \widehat{H}_t \right] \\ &= E^\pi \left[c_t(\widehat{X}_t, \pi_t(\widehat{H}_t), \Theta) + E^\pi \left[\sum_{s=t+1}^T c_s(\widehat{X}_s, \pi_s(\widehat{X}_{t+1}, \dots, \widehat{X}_s), \Theta) \mid \widehat{H}_{t+1} \right] \mid \widehat{H}_t \right], \\ &= E^\pi [c_t(\widehat{X}_t, \pi_t(\widehat{H}_t), \Theta) + v_{t+1}^\pi(\widehat{H}_{t+1}) \mid \widehat{H}_t], \end{aligned}$$

and so

$$v_t^\pi(h_t) = E^\pi [c_t(x_t, \pi_t(h_t), \Theta) + v_{t+1}^\pi(h_t, \widehat{X}_{t+1}) \mid \widehat{H}_t = h_t].$$

Next, by Lemma 2.4, we have

$$v_t^\pi(h_t) = \int_{\Theta} \int_{\mathcal{X}} (c_t(x_t, \pi_t(h_t), \theta) + v_{t+1}^\pi(h_t, x_{t+1})) P_{\theta, t+1}^{\pi^t, h_t}(dx_{t+1}) \xi_t^{\pi^t, h_t}(d\theta), \quad (4.6)$$

and by taking into account the form of $\widehat{\rho}_t$ and σ_t as in (3.19) and (3.20), respectively, we obtain (4.4). Finally, (4.5) is a direct consequence of (4.3) and Lemma 2.4. \square

In the case of the risk-sensitive rewards, that is Example 3.17, the recursiveness of ρ can be demonstrated in a way analogous to the above.

5 Risk-Averse Control Problem

Let v_1^π be as in (4.1). The control problem is to find

$$\min_{\pi \in \Pi} v_1^\pi(h_1), \quad (5.1)$$

as well as the optimal policy, say π^* , for which $v_1^{\pi^*}(h_1) = \min_{\pi \in \Pi} v_1^\pi(h_1)$. Note that given our set-up, an optimal policy does exist because the set Π is finite. However, we are interested in seeking an optimal policy in the class of quasi-Markov policies.

Definition 5.1. A policy $\pi \in \Pi$ is *quasi-Markov (QMP)* if

$$\pi_t(h_t) = \phi_t(x_t, \xi_t^{\pi, h_t})$$

for some function $\phi_t : \mathcal{X} \times \Theta \rightarrow \mathcal{U}$, $t = 1, \dots, T$.

5.1 The Bayes Operator

At each time t and for every policy π and history h_t , the measure $P_{t+1}^{\pi^t, h_t}$ (cf. (2.14)) describes the conditional joint distribution of the pair $(\widehat{X}_{t+1}, \Theta)$ in $\mathcal{X} \times \Theta$.

This measure admits two natural disintegrations. One of them is already obtained from (2.7), repeated here :

$$\begin{aligned} P_{t+1}^{\pi^t, h_t}(B \times D) &= P_{t+1, T}^{\pi^t, h_t}(B \times \mathcal{X}^{T-t-1} \times D) = \int_D P_{\theta, t+1, T}^{\pi^t, h_t}(B \times \mathcal{X}^{T-t-1}) \xi_t^{\pi, h_t}(d\theta) \\ &= P^\pi[\widehat{X}_{t+1} \in B, \Theta \in D \mid \widehat{H}_t = h_t], \end{aligned}$$

where $\xi_t^{\pi, h_t} \in \mathcal{P}(\Theta)$, is given as (cf. (2.8) and (2.10)) $\xi_t^{\pi, h_t}(D) = P^\pi[\Theta \in D \mid \widehat{H}_t = h_t] = P_{t+1, \Theta}^{\pi^t, h_t}(D)$. One can also disintegrate $P_{t+1}^{\pi^t, h_t}$ into its marginal on \mathcal{X} , say⁷ $P_{t+1, X}^{\pi^t, h_t}$, and the corresponding stochastic kernel, say $P_{t+1|X}^{\pi^t, h_t}$ from \mathcal{X} to Θ . That is, for any $B \times D \subset \mathcal{X} \times \Theta$,

$$\begin{aligned} P_{t+1}^{\pi^t, h_t}(B \times D) &= (P_{t+1, X}^{\pi^t, h_t} \otimes P_{t+1|X}^{\pi^t, h_t})(B \times D), \\ &= \int_B P_{t+1|x}^{\pi^t, h_t}(D) P_{t+1, X}^{\pi^t, h_t}(dx) \\ &= P^\pi[\widehat{X}_{t+1} \in B, \Theta \in D \mid \widehat{H}_t = h_t], \end{aligned} \tag{5.2}$$

where we used the simplified notation $P_{t+1|X}^{\pi^t, h_t}(D)$ for $P_{t+1|X}^{\pi^t, h_t}(x, D)$.

The kernel $P_{t+1|x}^{\pi^t, h_t}$ is the Bayes operator which describes the dynamics of the belief states, as documented in the next result.

Lemma 5.2. *For $t = 1, \dots, T-1$, $h_t \in H_t$, $x_{t+1} \in \mathcal{X}$ and $D \subset \Theta$, we have*

$$\begin{aligned} \xi_{t+1}^{\pi, (h_t, x_{t+1})}(D) &= P_{t+1|x_{t+1}}^{\pi^t, h_t}(D) \\ &= \int_D \frac{K_\theta(x_{t+1}|x_t, \pi_t(h_t))}{P_{t+1}^{\pi^t, h_t}[\{x_{t+1}\} \times \Theta]} \xi_t^{\pi, h_t}(d\theta), \end{aligned} \tag{5.3}$$

where

$$\xi_1^{\pi, x_1}(\theta) = \xi_1(\theta).$$

Proof. First, note that

$$P_{t+1, X}^{\pi^t, h_t}(B) = P^\pi[\widehat{X}_{t+1} \in B \mid \widehat{H}_t = h_t]. \tag{5.4}$$

Take $B = \{x_{t+1}\}$. Then, using (5.2) and (5.4), we obtain

$$P^\pi[\widehat{X}_{t+1} = x_{t+1}, \Theta \in D \mid \widehat{H}_t = h_t] = P_{t+1|x_{t+1}}^{\pi^t, h_t}(D) P^\pi[\widehat{X}_{t+1} = x_{t+1} \mid \widehat{H}_t = h_t],$$

and thus

$$\begin{aligned} P_{t+1|x_{t+1}}^{\pi^t, h_t}(D) &= \frac{P^\pi[\widehat{X}_{t+1} = x_{t+1}, \Theta \in D \mid \widehat{H}_t = h_t]}{P^\pi[\widehat{X}_{t+1} = x_{t+1} \mid \widehat{H}_t = h_t]} \\ &= P^\pi[\Theta \in D \mid \widehat{H}_{t+1} = (h_t, x_{t+1})] = \xi_{t+1}^{\pi, (h_t, x_{t+1})}(D), \end{aligned}$$

⁷For simplicity of notations, we write $P_{t+1, X}^{\pi^t, h_t}$ instead of more coherent notation $P_{t+1, X_{t+1}}^{\pi^t, h_t}$. Similar remark applies to the kernel $P_{t+1|X}^{\pi^t, h_t}$.

which proves the first equality in (5.3). The second one follows from the following chain of equalities,

$$\begin{aligned}\xi_{t+1}^{\pi, (h_t, x_{t+1})}(\theta) &= P^\pi[\Theta = \theta \mid \widehat{H}_{t+1} = (h_t, x_{t+1})] \\ &= P^\pi[\Theta = \theta \mid \widehat{H}_t = h_t] \frac{P^\pi[\widehat{X}_{t+1} = x_{t+1} \mid \Theta = \theta, \widehat{H}_t = h_t]}{P^\pi[\widehat{X}_{t+1} = x_{t+1} \mid \widehat{H}_t = h_t]} \\ &= \xi_t^{\pi, h_t}(\theta) \frac{K_\theta(x_{t+1} \mid x_t, \pi_t(h_t))}{P_{t+1}^{\pi_t, h_t}[\{x_{t+1}\} \times \Theta]},\end{aligned}$$

where in last equality we used (2.6) and that $P_\theta^\pi(B) = P^\pi(B \mid \Theta = \theta)$. \square

5.2 Optimal control problem corresponding to Example 3.15

In this section we will study the optimal control problem corresponding to the Example 3.15 classical additive reward case, that will serve as the base for the general case. In what follows, we denote by (x, ξ) an element of the set $\mathcal{X} \times \mathcal{P}(\Theta)$.

Recall (4.3). Accordingly, we have for $t = T$

$$\begin{aligned}v_T^\pi(h_T) &= \rho_{T,T} \left(c_T(x_T, \pi_T(h_T), \cdot), P_{T+1,T}^{\pi_T, h_T} \right) \\ &= \int_{\Theta} c_T(x_T, \pi_T(h_T), \theta) P_{T+1,T}^{\pi_T, h_T}(d\theta) \\ &= \int_{\Theta} c_T(x_T, \pi_T(h_T), \theta) \xi_T^{\pi, h_T}(d\theta) \\ &= E^\pi \left(c_T(x_T, \pi_T(h_T), \Theta) \mid \widehat{H}_T = h_T \right).\end{aligned}$$

Thus, observing that ξ_T^{π, h_T} , does not depend on π_T , letting $x_T = x$ and $\xi_T^{\pi, h_T} = \xi$, we compute the candidate-optimal quasi-Markov control ϕ_T as

$$\phi_T(x, \xi) = \arg \min_{u \in \mathcal{U}} \int_{\Theta} c_T(x, u, \theta) \xi(d\theta).$$

We define the Bellman function at time $t = T$:

$$V_T(x, \xi) = \min_{u \in \mathcal{U}} \int_{\Theta} c_T(x, u, \theta) \xi(d\theta) = \int_{\Theta} c_T(x, \phi_T(x, \xi), \theta) \xi(d\theta).$$

Now, we proceed to time $t = T - 1$. Noting that $\xi_{T-1}^{\pi, h_{T-1}}$, does not depend on π_{T-1} , letting $x_{T-1} = x$ and $\xi_{T-1}^{\pi, h_{T-1}} = \xi$, we compute the candidate-optimal quasi-Markov control ϕ_{T-1} as

$$\phi_{T-1}(x, \xi) = \arg \min_{u \in \mathcal{U}} \int_{\Theta} \left(c_{T-1}(x, u, \theta) + \int_{\mathcal{X}} V_T(x_T, \tilde{\xi}_T^{u, x_T, \xi}) K_\theta(dx_T \mid x, u) \right) \xi(d\theta), \quad (5.5)$$

where (cf. (5.3))

$$\tilde{\xi}_T^{u, x_T, \xi}(\theta) = \xi(\theta) \frac{K_\theta(x_T \mid x, u)}{\int_{\Theta} K_\theta(x_T \mid x, u) \xi(d\theta)}. \quad (5.6)$$

The corresponding Bellman function is

$$\begin{aligned}V_{T-1}(x, \xi) &= \min_{u \in \mathcal{U}} \int_{\Theta} \left(c_{T-1}(x, u, \theta) + \int_{\mathcal{X}} V_T(x_T, \tilde{\xi}_T^{u, x_T, \xi}) K_\theta(dx_T \mid x, u) \right) \xi(d\theta) \\ &= \int_{\Theta} \left(c_{T-1}(x, \phi_{T-1}(x, \xi), \theta) + \int_{\mathcal{X}} V_T(x_T, \tilde{\xi}_T^{\phi_{T-1}(x, \xi), x_T, \xi}) K_\theta(dx_T \mid x, \phi_{T-1}(x, \xi)) \right) \xi(d\theta).\end{aligned}$$

Following this pattern, we arrive at the dynamic programming (DP) backward recursion:

$$V_t(x, \xi) = \min_{u \in \mathcal{U}} \int_{\Theta} \left(c_t(x, u, \theta) + \int_{\mathcal{X}} V_{t+1}(x_{t+1}, \tilde{\xi}_{t+1}^{u, x_{t+1}, \xi}) K_{\theta}(dx_{t+1}|x, u) \right) \xi(d\theta), \quad t \in \mathcal{T}, \quad (5.7)$$

where (cf. (5.3))

$$\tilde{\xi}_{t+1}^{u, x_{t+1}, \xi}(\theta) = \xi(\theta) \frac{K_{\theta}(x_{t+1}|x, u)}{\int_{\Theta} K_{\theta}(x_{t+1}|x, u) \xi(d\theta)}, \quad (5.8)$$

and

$$V_{T+1} \equiv 0. \quad (5.9)$$

Note that (5.7) is a counterpart of (4.4).

Accordingly, for $t = 1, \dots, T$ we define the candidate-optimal quasi-Markov control ϕ_t as

$$\phi_t(x, \xi) = \arg \min_{u \in \mathcal{U}} \int_{\Theta} \left(c_t(x, u, \theta) + \int_{\mathcal{X}} V_{t+1}(x_{t+1}, \tilde{\xi}_{t+1}^{u, x_{t+1}, \xi}) K_{\theta}(dx_{t+1}|x, u) \right) \xi(d\theta).$$

Recall that ξ_1 is a given prior distribution for Θ . Also, recall that $h_1 = x_1$.

Next, define a policy π^* as follows,

$$\begin{aligned} \pi_1^*(h_1) &= \phi_1(x_1, \xi_1) \\ \pi_t^*(h_t) &= \phi_t(x_t, \widehat{\xi}_t^{\pi^*, h_t}), \quad t = 2, \dots, T, \end{aligned} \quad (5.10)$$

where

$$\widehat{\xi}_2^{\pi^*, h_2} = \widehat{\xi}_2^{\pi_1^*(h_1), x_2, \xi_1}, \quad \widehat{\xi}_3^{\pi^*, h_3} = \widehat{\xi}_3^{\pi_2^*(h_2), x_3, \widehat{\xi}_2^{\pi^*, h_2}}, \dots \quad (5.11)$$

The next result is the optimality verification theorem.

Theorem 5.3. *We have,*

$$\min_{\pi \in \Pi} v_1^{\pi}(h_1) = v_1^{\pi^*}(h_1) = V_1(x_1, \xi_1).$$

Proof. Let $\pi \in \Pi$. For $t = T$ we have

$$v_T^{\pi}(h_T) \geq V_T(x_T, \xi_T^{\pi, h_T}) = \int_{\Theta} c_T(x_T, \phi_T(x_T, \xi_T^{\pi, h_T}), \theta) \xi_T^{\pi, h_T}(d\theta).$$

For $t = T - 1$, using the above, the recursion (4.6), and (5.3), we have

$$\begin{aligned} v_{T-1}^{\pi}(h_{T-1}) &= \int_{\Theta} \left(c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-1}), \theta) + \int_{\mathcal{X}} v_T^{\pi}(h_T) K_{\theta}(dx_T|x_{T-1}, \pi_{T-1}(h_{T-1})) \right) \xi_{T-1}^{\pi, h_{T-1}}(d\theta) \\ &\geq \int_{\Theta} \left(c_{T-1}(x_{T-1}, \pi_{T-1}(h_{T-1}), \theta) + \int_{\mathcal{X}} V_T(x_T, \xi_T^{\pi, h_T}) K_{\theta}(dx_T|x_{T-1}, \pi_{T-1}(h_{T-1})) \right) \xi_{T-1}^{\pi, h_{T-1}}(d\theta) \\ &\geq \int_{\Theta} \left(c_{T-1}(x_{T-1}, \phi_{T-1}(x_{T-1}, \xi_{T-1}^{\pi, h_{T-1}}), \theta) \right. \\ &\quad \left. + \int_{\mathcal{X}} V_T(x_T, \widehat{\xi}_T^{\phi_{T-1}(x_{T-1}, \xi_{T-1}^{\pi, h_{T-1}}), x_T, \xi_{T-1}^{\pi, h_{T-1}}} \right) K_{\theta}(dx_T|x_{T-1}, \phi_{T-1}(x_{T-1}, \xi_{T-1}^{\pi, h_{T-1}})) \xi_{T-1}^{\pi, h_{T-1}}(d\theta) \\ &= V_{T-1}(x_{T-1}, \xi_{T-1}^{\pi, h_{T-1}}). \end{aligned}$$

Likewise, for $t = 1, \dots, T - 2$, we have

$$\begin{aligned} v_t^\pi(h_t) &\geq V_t(x_t, \xi_t^{\pi, h_t}) = \int_{\Theta} \left(c_t(x_t, \phi_t(x_t, \xi_t^{\pi, h_t}), \theta) \right. \\ &\quad \left. + \int_{\mathcal{X}} V_{t+1}(x_{t+1}, \phi_{t+1}(x_{t+1}, \tilde{\xi}_{t+1}^{\phi_t(x_t, \xi_t^{\pi, h_t}), x_{t+1}, \xi_t^{\pi, h_t}})) K_\theta(dx_{t+1} | x_t, \phi_t(x_t, \xi_t^{\pi, h_t})) \right) \xi_t^{\pi, h_t}(d\theta). \end{aligned}$$

Now, if π and ξ_t^{π, h_t} , $t \in \mathcal{T}$, above are replaced with π^* and $\widehat{\xi}_t^{\pi^*, h_t}$, $t \in \mathcal{T}$, respectively, then the inequalities above become equalities, proving that π^* is an optimal strategy. \square

Recalling (3.19) and (3.20), we note that the key DP recursion (5.5) can be written as

$$V_t(x, \xi) = \min_{u \in \mathcal{U}} \widehat{\rho}_t \left(\left\{ c_t(x, u, \theta) + \sigma_t(V_{t+1}(\cdot, \tilde{\xi}_{t+1}^{u, \cdot, \xi}); K_\theta(x, u)), \theta \in \Theta \right\}; \xi \right),$$

subject to (5.8) and (5.9).

Example 5.4. We remark that the optimal control problem considered in this section can be cast in the classical optimal investment and consumption problem, now also subject to model uncertainty. Namely, consider an investor with initial capital \bar{x}_1 , who can invest in d assets, with \bar{X}_t denoting the portfolio value at time t , which of course is observed by the investor. The investor rebalances the portfolio at each time t , following a self-financing trading strategy (policy) π , that may satisfy additional trading constrains, such as short selling constrains, turn over constraints, etc. The investor is also allowed to consume at each time t part of the wealth, say z_t , that does not exceed \bar{X}_t . We postulate that the investor maximizes the expected utility of consumptions and terminal wealth using the utility functions V^β and U^γ , respectively, where $\beta, \gamma \in \mathbb{R}$ stand for risk aversion-parameters. We refer the reader to [BR17, Section 4] for detailed formulation of this problem in the MDP framework.

Additionally, we assume that the investor faces the Knightian uncertainty about the model of the underlying assets, described in terms of a (finite) parametric set $\mathbf{\Lambda} \subset \mathbb{R}^m$; see [BCC⁺19] for an overview of MDPs under Knightian uncertainty.

Moreover, we suppose that the investor is also uncertain about her risk aversion parameters $(\beta, \gamma) \in \mathbf{\Gamma} \subset \mathbb{R}^2$. We emphasize that this additional feature of an unknown risk aversion parameter is practically important. Generally speaking, it is difficult to determine the investor's risk aversion parameter, which is well documented in the behavioral finance literature. This becomes especially relevant in the context of fast-growing robo-advising industry that typically deals with unsophisticated investors, and which establishes investor's risk preferences without human intervention. At each time t , the investor reports through process Y_t her subjective degree of happiness about the performance of her investment. For example, one can take Y_t to be a Bernoulli random variable with $Y_t = 1$ corresponding to happy and $Y_t = 0$ meaning unhappy about her investment, and then follow a similar setup to the clinical trials Example 3.6 and incorporate the uncertainty about (β, γ) into the original MDP formulation.

Now we consider the observed state process $X_t = (\bar{X}_t, Y_t)$, and we take $\theta = (\beta, \gamma, \lambda) \in \Theta = \mathbf{\Gamma} \times \mathbf{\Lambda}$ representing the model uncertainty in this model. Consequently, we define the cost functionals

$$\begin{aligned} c_t &= V^\beta(z_t(\bar{x}_t, \pi_t)) + F(y_t, \pi_t, \beta, \gamma), \quad t = 1, \dots, T - 1, \\ c_T &= V^\beta(z_T(\bar{x}_T, \pi_T)) + U^\gamma(\bar{x}_T), \end{aligned}$$

where F is a penalty for ‘deviating’ from the true risk-aversion parameters. Using the expectation as the risk functional (cf. Example 3.15), the problem (5.1) becomes the optimal investment and consumption problem. Theorem 5.3 gives the solution to this problem. Detailed model specification and analysis is beyond the scope of this work and it will be addressed in future works.

5.3 Optimal control problem corresponding to Example 3.17

We will present the solution to the optimal control problem for the clinical trials example with the risk-sensitive criterion. Namely, for $t \in \mathcal{T}$, we consider

$$\begin{aligned} v_t^\pi(h_t) &= \rho_t \left(c_t(x_t, \pi_t(h_t), \cdot), c_{t+1}(\cdot, \pi_{t+1}(h_t, \cdot), \cdot), \right. \\ &\quad \left. c_{t+2}(\cdot, \pi_{t+2}(h_t, \cdot, \cdot), \cdot), \dots, c_T(\cdot, \pi_T(h_t, \cdot, \dots, \cdot), \cdot), P_{t+1,T}^{\pi_t, h_t} \right) \\ &= \frac{1}{\varkappa} \ln E^\pi \left(\exp \left(\varkappa \left(c_t(x_t, \pi_t(h_t), \Theta) + \sum_{k=t+1}^T c_k(\widehat{X}_k, \pi_k(h_t, \widehat{X}_{t+1}, \dots, \widehat{X}_k), \Theta) \right) \right) \middle| \widehat{H}_t = h_t \right) \\ &= \frac{1}{\varkappa} \ln w_t^\pi(h_t). \end{aligned}$$

It is clear that problem (5.1) is equivalent to the following problem

$$\min_{\pi \in \Pi} w_1^\pi(h_1),$$

For $t = T$ we have

$$w_T^\pi(h_T) = E^\pi \left(\exp(\varkappa c_T(x_T, \pi_T(h_T), \Theta)) \middle| \widehat{H}_T = h_T \right) = \int_{\Theta} e^{\varkappa c_T(x_T, \pi_T(h_T), \theta)} \xi_T^{\pi, h_T}(d\theta).$$

As above, we denote by (x, ξ) an element of the set $\mathcal{X} \times \mathcal{P}(\Theta)$. Thus, observing that ξ_T^{π, h_T} does not depend on π_T , and letting $x_T = x$ and $\xi_T = \xi$, we compute the candidate optimal quasi-Markov control φ_T as

$$\varphi_T(x, \xi) = \arg \min_{u \in \mathcal{U}} \int_{\Theta} e^{\varkappa c_T(x, u, \theta)} \xi(d\theta).$$

We define the Bellman function at time $t = T$ as

$$W_T(x, \xi) = \min_{u \in \mathcal{U}} \int_{\Theta} e^{\varkappa c_T(x, u, \theta)} \xi(d\theta) = \int_{\Theta} e^{\varkappa c_T(x, \varphi_T(x, \xi), \theta)} \xi(d\theta).$$

Now, we proceed to time $t = T - 1$. Given $x_{T-1} = x$ and $\xi_{T-1} = \xi$ we compute the candidate optimal quasi-Markov control φ_{T-1} as

$$\varphi_{T-1}(x, \xi) = \arg \min_{u \in \mathcal{U}} \int_{\Theta} \int_{\mathcal{X}} e^{\varkappa c_{T-1}(x, u, \theta)} W_T(x_T, \widetilde{\xi}_T^{u, x_T, \xi}) K_\theta(dx_T | x, u) \xi(d\theta), \quad (5.12)$$

where $\widetilde{\xi}_T^{u, x_T, \xi}$ is given by (5.6). The corresponding Bellman function is

$$\begin{aligned} W_{T-1}(x, \xi) &= \min_u \int_{\Theta} \int_{\mathcal{X}} e^{\varkappa c_{T-1}(x, u, \theta)} W_T(x_T, \widetilde{\xi}_T^{u, x_T, \xi}) K_\theta(dx_T | x, u) \xi(d\theta) \\ &= \int_{\Theta} \int_{\mathcal{X}} e^{\varkappa c_{T-1}(x, \varphi_{T-1}(x, \xi), \theta)} W_T(x_T, \widetilde{\xi}_T^{\varphi_{T-1}(x, \xi), x_T, \xi}) K_\theta(dx_T | x, \varphi_{T-1}(x, \xi)) \xi(d\theta). \end{aligned}$$

Following this pattern, we arrive at the DP backward recursion:

$$W_t(x, \xi) = \min_u \int_{\Theta} \int_{\mathcal{X}} e^{\varkappa c_t(x, u, \theta)} W_{t+1}(x_{t+1}, \widetilde{\xi}_{t+1}^{u, x_{t+1}, \xi}) K_\theta(dx_{t+1} | x, u) \xi(d\theta), \quad t \in \mathcal{T}, \quad (5.13)$$

where as in the previous example $\widetilde{\xi}_{t+1}^{u, x_{t+1}, \xi}(\{\theta\})$ is given by (5.8). and $W_{T+1} \equiv 1$. Note that (5.13) is a counterpart of (4.4).

Accordingly, for $t \in \mathcal{T}$, we define the candidate-optimal quasi-Markov control φ_t as

$$\varphi_t(x, \xi) = \arg \min_u \int_{\Theta} \int_{\mathcal{X}} e^{\varkappa c_t(x, u, \theta)} W_{t+1}(x_{t+1}, \tilde{\xi}_{t+1}^{u, x_{t+1}, \xi}) K_{\theta}(dx_{t+1}|x, u) \xi(d\theta),$$

with ξ_1 being the given prior distribution for Θ , and $h_1 = x_1$.

The policy π^* is defined by analogy to (5.10). The following verification theorem can be proved in a way analogous to the proof of Theorem 5.3, so we skip its proof.

Theorem 5.5. *The following hold true*

$$\min_{\pi \in \Pi} v_1^{\pi}(h_1) = v_1^{\pi^*}(h_1) = \frac{1}{\varkappa} \ln(W_1(x_1, \xi_1)).$$

We emphasise that the key DP recursion (5.12) may be written as

$$W_t(x, \xi) = \min_{u \in \mathcal{U}} \hat{\rho}_t \left(\{c_t(x, u, \theta) + \sigma_t(W_{t+1}(\cdot, \tilde{\xi}_{t+1}^{u, \cdot, \xi}); K_{\theta}(x, u)), \theta \in \Theta\}; \xi \right),$$

where for a function f on Θ , $\xi \in \mathcal{P}(\Theta)$, and a function h on \mathcal{X} , we have

$$\hat{\rho}_t(\{f(\theta), \theta \in \Theta\}; \xi) = \int_{\Theta} e^{\varkappa f(\theta)} \xi(d\theta),$$

and where

$$\sigma_t(h; K_{\theta}(x, u)) = \frac{1}{\varkappa} \ln \int_{\mathcal{X}} h(x_{t+1}) K_{\theta}(dx_{t+1}|x, u).$$

5.4 Solution of the optimal control problem for general recursive risk filters

Let ρ be a recursive risk filter, and let

$$\begin{aligned} v_t^{\pi}(h_t) = & \rho_t \left(c_t(x_t, \pi_t(h_t), \cdot), c_{t+1}(\cdot, \pi_{t+1}(h_t, \cdot), \cdot), c_{t+2}(\cdot, \pi_{t+2}(h_t, \cdot, \cdot), \cdot), \dots \right. \\ & \left. \dots, c_T(\cdot, \pi_T(h_t, \cdot, \dots, \cdot), \cdot), P_{t+1, T}^{\pi_t, h_t} \right), \quad t \in \mathcal{T}. \end{aligned} \quad (5.14)$$

Consider the general problem (5.1) with $v_t^{\pi}(h_t)$ as in (5.14).

Using reasoning analogous to the one employed in Sections 5.2 and 5.3 one can prove the following result, proof of which we omit here.

Theorem 5.6. *There exist operators $\hat{\rho}_t$ and σ_t , $t \in \mathcal{T}$, and a function V^* such that for the functions v_t^* defined recursively as*

$$\begin{aligned} v_{T+1}^*(x) &= V^*(x), \quad x \in \mathcal{X}, \\ v_t^*(x, \xi) &= \min_{u \in \mathcal{U}} \hat{\rho}_t \left(\{c_t(x, u, \theta) + \sigma_t(v_{t+1}^*(\cdot, \tilde{\xi}_{t+1}^{u, \cdot, \xi}); K_{\theta}(x, u)), \theta \in \Theta\}; \xi \right), \\ & \quad t = T - 1, \dots, 1, \quad x \in \mathcal{X}, \quad \xi \in \mathcal{P}(\Theta), \end{aligned}$$

subject to

$$\tilde{\xi}_{t+1}^{u, x', \xi}(\theta) = \xi(\theta) \frac{K_{\theta}(x'|x, u)}{\int_{\Theta} K_{\theta}(x'|x, u) \xi(d\theta)}, \quad t \in \mathcal{T}, \quad x, x' \in \mathcal{X}, \quad \xi \in \mathcal{P}(\Theta),$$

we have that

$$\min_{\pi \in \Pi} v_1^{\pi}(h_1) = v_1^*(x_1, \xi_1).$$

Moreover, the policy π^* defined as in (5.10) and (5.11), with the ϕ_t 's given as

$$\phi_t(x, \xi) = \arg \min_{u \in \mathcal{U}} \widehat{\rho}_t \left(\left\{ c_t(x, u, \theta) + \sigma_t(v_{t+1}^*(\cdot, \widetilde{\xi}_{t+1}^{u, \cdot; \xi}); K_\theta(x, u)), \theta \in \Theta \right\}; \xi \right),$$

$$t = 1, \dots, T-1, \quad x \in \mathcal{X}, \quad \xi \in \mathcal{P}(\Theta),$$

is an optimal policy, that is

$$\min_{\pi \in \Pi} v_1^\pi(h_1) = v_1^{\pi^*}(h_1).$$

The form of the operators $\widehat{\rho}_t$ and σ_t , $t \in \mathcal{T}$, depends on the form of ρ , and it can be explicitly written in terms of ρ .

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References

- [BCC⁺19] T. R. Bielecki, T. Chen, I. Cialenco, A. Cousin, and M. Jeanblanc. Adaptive robust control under model uncertainty. *SIAM Journal on Control and Optimization*, 57(2):925–946, 2019.
- [BR14] N. Bäuerle and U. Rieder. More Risk-Sensitive Markov Decision Processes. *Mathematics of Operations Research*, 39(1):105–120, feb 2014.
- [BR17] N. Bäuerle and U. Rieder. Zero-sum risk-sensitive stochastic games. *Stochastic Processes and their Applications*, 127(2):622 – 642, 2017.
- [DL14] M. Davis and S. Lleo. *Risk-Sensitive Investment Management*, volume 19 of *Advanced Series on Statistical Science & Applied Probability*. World Sci., 2014.
- [FR18] J. Fan and A. Ruszczyński. Risk measurement and risk-averse control of partially observable discrete-time Markov systems. *Math. Methods Oper. Res.*, 88(2):161–184, 2018.
- [FR22] J. Fan and A. Ruszczyński. Process-based risk measures and risk-averse control of discrete-time systems. *Math. Program.*, 191(1, Ser. B):113–140, 2022.
- [LRZ21] Y. Lin, Y. Ren, and E. Zhou. A Bayesian risk approach to MDPs with parameter uncertainty. *Preprint arXiv:2106.02558*, 2021.
- [LS20] T. Lattimore and C. Szepesvári. *Bandit algorithms*. Cambridge University Press, 2020.
- [SB18] R. S. Sutton and A. G. Barto. *Reinforcement learning: An introduction*. MIT press, 2018.