Controlled Markov processes and mathematical finance

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Abstract

These lectures are concerned with optimal control of Markov diffusion processes with complete state information, and with some applications in financial economics. Problems on a finite time horizon, and on an infinite horizon with discounted cost or ergodic (average cost per unit time) criterion are considered. We also consider risk sensitive stochastic control on an infinite horizon, with expected exponential-of-integral cost criteria. These problems are related, through a logarithmic transformation, to infinite horizon ergodic stochastic control and stochastic differential games. Risk sensitive control provides a link between stochastic and deterministic (robust control) approaches to disturbance attenuation. This is done by considering the robust control model as a small noise intensity limit of a corresponding risk sensitive model. In mathematical finance, we consider some models of optimal portfolio choice which are extensions of the classical Merton model. Problems of optimal long-term growth of expected utility of wealth are reformulated as infinite horizon risk sensitive control problems. Explicit solutions are given in absence of investment control constraints. In addition, a robust control approach to the Black-Scholes formula for pricing stock options is mentioned.

Introduction

These notes are based on a series of five lectures, with the same titles as chapters 1 through 5 which follow.

The lectures were intended to be introductory, rather than a definitive treatment of the best known results. Proofs are outlined only for rather easy results (for instance, Verification Theorems in stochastic control), with references to deeper results in the literature. In particular, the reader may consult several standard books on stochastic control of Markov diffusions cited in the Reference list.

We limit the discussion to optimal control of Markov diffusion processes, with state dynamics governed by Ito-sense stochastic differential equations. Moreover, we assume that at each instant of time the controller knows the system state. We follow the dynamic programming method. With this method the value function has a key role. For controlled Markov diffusions, it satisfies a second order, nonlinear partial differential equation (PDE) of parabolic or elliptic type. For nondegenerate controlled Markov diffusions the dynamic programming PDE, with appropriate boundary conditions, turns out to have solutions which

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are smooth (classical solutions). Without the nondegeneracy assumption, the value function satisfies this PDE in the viscosity sense.

The literature on stochastic models in mathematical finance is by now quite large. We have chosen to discuss only a few models which illustrate the role of stochastic control. In particular, risk sensitive control arises naturally in considering certain long-term growth questions for investment portfolios. The role of stochastic control in studying macroeconomic growth/debt models has been much less explored. Some preliminary thoughts in that direction are mentioned in section 2.6.

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1 Controlled Markov processes

1.1 Introduction

Following Fleming-Soner [FlS, Chap. 3] let us outline formally a general framework for optimal control of Markov processes with complete state observations. This introduction is purely heuristic. No attempt is made at this level of generality to state precise assumptions or theorems. However, things will be stated more precisely when we specialize to controlled Markov diffusions, beginning in section 1.2.

Let x_t denote the state of the system being controlled, and u_t the control at time t. We have $x_t \in \Sigma$, $u_t \in U$, where Σ, U denote respectively the state space and control space. We consider either $0 \leq t \leq T$ (fixed finite time horizon) or $0 \leq t < \infty$ (infinite time horizon). Also, we discuss briefly in section 1.4 the problem of controlling x_t up to the time when it exits from some region $\mathcal{O}$.

For fixed finite horizon, consider a cost criterion of the form

$$\mathcal{J} = \int_0^T L(x_t, u_t)dt + g(x_T), \tag{1.1}$$

where $L(x, u)$ is a running cost function and $g(x)$ a terminal cost function. For infinite time horizon, we consider the discounted cost criterion

$$\mathcal{J} = \int_0^\infty e^{-\beta t} L(x_t, u_t)dt, \quad \beta > 0. \tag{1.2}$$

In chapter 3 ergodic cost criteria are considered.

A traditional stochastic control objective is to maximize (or minimize) $J = E(\mathcal{J})$, and this is the objective which we now consider. In chapter 4 we consider instead a risk sensitive control objective, namely to maximize $E[F(\mathcal{J})]$, where F is an increasing function which is either convex or concave. An important example is an exponential

$$F(\mathcal{J}) = \exp(\theta \mathcal{J}), \quad \theta > 0$$

In the general formulation for controlled Markov processes [FlS, Chap. 3] a family of linear operators G^u is given ($u \in U$). Since the present discussion is merely heuristic, we will not be precise about the space on which G^u operates, nor the domain of G^u (which is an unbounded operator for controlled Markov diffusions). However, the idea is that if the control is fixed ($u_t \equiv u$ for all t), then x_t is a Markov process and G^u is its generator. Moreover, if $E_x\phi(x_t)$