Chapter 11
Stopping Problems in Finance

Typical stopping problems in finance involve the *pricing of American options*. It can be shown by using no-arbitrage arguments that the price of an American option is the value of an optimal stopping problem under a risk neutral probability measure and the optimal stopping time is the optimal exercise time of the option. In order to have a complete financial market without arbitrage we restrict the first section on pricing American options to the binomial model. An algorithm is presented for pricing American options and the American put option is investigated in detail. In particular also perpetual American put options are studied. In Section 11.2 so-called *credit granting* problems are considered. Here the decision maker has to decide whether or not a credit is extended. In this context, a Bayesian Model is also presented.

11.1 Pricing of American Options

A classical application of optimal stopping problems in finance are American options: In order to find the fair price of an American option and its optimal exercise time, we have to solve an optimal stopping problem with finite horizon N. In contrast to a European option, the buyer of an American option can choose to exercise any time up to and including the expiration time N. In what follows we will consider the *binomial model* as underlying financial market (see Section 3.1) with the assumption $\boldsymbol{d} < 1+i < \boldsymbol{u}$ which implies no arbitrage opportunities and the existence of a unique equivalent martingale measure $\mathbb{Q}$. This measure $\mathbb{Q}$ is used for pricing and is also called risk neutral probability measure. Under $\mathbb{Q}$, the probability for an up movement of the stock is given by

$$q = \frac{1+i-\boldsymbol{d}}{\boldsymbol{u}-\boldsymbol{d}}. \tag{11.1}$$

We will first consider general American options with finite expiration date and concentrate on the case of path-independent options.

N. Bäuerle and U. Rieder, *Markov Decision Processes with Applications to Finance*, Universitext, DOI 10.1007/978-3-642-18324-9_11,

American Options

We concentrate our analysis on path-independent American options, i.e. the process (X_n) which has to be stopped is given by the stock price process $X_n = S_n$ itself. A path-independent American option yields the payoff $h(S_n)$ if it is exercised at time n, i.e. the payoff function depends only on the current stock price and not on its path. The expiration date is assumed to be N. However, the option may never be exercised in which case the payoff is zero. Thus, it is easy to see that it cannot be optimal to exercise when $h(S_n) < 0$ and we can equivalently choose $h^+(S_n)$ as a payoff. Let us denote $\beta := (1+i)^{-1}$. The price of this option at time zero is then computed as

$$\sup_{\tau \le N} \mathbb{E}_x^{\mathbb{Q}} \left[\beta^\tau h^+(S_\tau)\right]$$

where the supremum is taken over all stopping times τ with $\mathbb{P}(\tau \le N) = 1$. $S_0 = x$ is the stock price at time zero and the expectation is taken with respect to the risk neutral measure $\mathbb{Q}$. For example in the case of a European put option with strike price K, the payoff function h is given by $h(x) = K - x$. This stopping problem can be formulated as a stationary Markov Decision Problem (see Section 10.1). The data of the stopping problem is thus:

- $E := \mathbb{R}_+$, where x denotes the current stock price,
- $A := \{0, 1\}$ where $a = 0$ means continue and $a = 1$ means exercise,
- $Q^X(B|x) = q\delta_{x\boldsymbol{u}}(B) + (1-q)\delta_{x\boldsymbol{d}}(B), x \in E$ for Borel sets B where q is given by (11.1),
- $g(x) := h^+(x)$ and $c(x) \equiv 0$,
- $\beta := (1+i)^{-1} \in (0, 1]$ is the discount factor.

Note that when S_0 is the initial stock price, then at time n in the binomial model the only possible stock prices are given by

$$\{S_0 \boldsymbol{u}^k \boldsymbol{d}^{n-k} \mid k = 0, \dots, n\}.$$

However, it is sometimes convenient to choose a continuous state space. Assumption (B_N) is satisfied since

$$\sup_{n \le \tau \le N} \mathbb{E}_{nx}^{\mathbb{Q}} \left[\sum_{k=n}^{\tau-1} \beta^k c^+(X_k) + \beta^\tau h^+(X_\tau)\right] \le \mathbb{E}_{nx}^{\mathbb{Q}} \left[\sum_{k=n}^{N} h^+(X_k)\right] < \infty$$

because X_k can only take a finite number of possible values with positive probability for all $k = 1, \dots, N$. Moreover, the following value iteration holds for this problem (cf. Theorem 10.1.5).