Markov Chains and Applications in Finance

Yining Guo

15/09/2020

Transition Probability

- Mouse in a Maze.

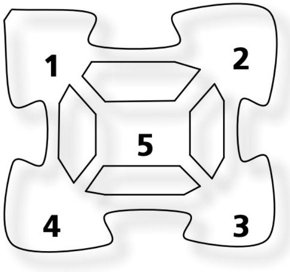

Figure 1: The Maze

The mouse can only go to adjacent rooms;
its decisions are independent of any history and only depend on the current room the mouse is in;
the mouse cannot stay in the same room forever.

Transition Probability

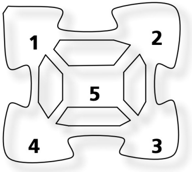

Figure 1: The Maze

- Let P_{ji} be the probability of going to room j when the mouse is in room i by one move.
- $P_{21} = \frac{1}{3}$ is the probability of moving to room 2 from room 1.
- $P_{11} = 0$ is the probability of staying at room 1 forever.

Matrix Representations

- To make it more computationally efficient: use matrix representations

  Transition Matrix: $P = \begin{bmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \ldots & \mathrm{P}_{1r} \\ . & . & . & \\ . & . & . & \\ . & . & . & \\ & & & \\ \mathrm{P}_{r1} & \mathrm{P}_{r2} & \ldots & \mathrm{P}_{rr} \end{bmatrix}$

- To find the transition probability from state i to state j: look up P_{ji} in the matrix.

Transition Probability

- What is the probability of getting to room 5 after 1 step? The starting state matters.
- If we flip a coin to determine the starting point of the mouse: say, we put the mouse in room 1 if we get a head, and put the mouse in room 2 if we get a tail.
- P(starting from room 1) $=$ P(starting from room 2) $= \frac{1}{2}$
- P(ending up in room 5) $= \frac{1}{2}P_{51} + \frac{1}{2}P_{52}$

Distributions of Transition Probabilities

- Define every room as a state. What if the probabilities of initial states are different? Say, roll an unfair 5-sided dice to determine the starting room.
- $S = \begin{bmatrix} \mathrm{q}_1 \\ \mathrm{q}_2 \\ \mathrm{q}_3 \\ \mathrm{q}_4 \\ \mathrm{q}_5 \end{bmatrix}$ represents the starting state of the mouse.
- $S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ means that the mouse is in room 1 at the beginning in the fair dice case.

- Combining the transition matrix and the distribution of probabilities

  1. The distribution of transition probabilities on the states after 1 step starting from state s is the matrix-vector product
  $(Pq)_i = \sum_{j=1}^{r} P_{ij} q_j = P_{i1} q_1 + P_{i2} q_2 + ... + P_{ir} q_r$

  2. The probability of ending up in state j after 2 steps starting from state i is:
  $(P^2)_{ji} = \sum_{k=1}^{r} P_{jk} P_{ki}$

Markov Chains

- In English,
  [Next State] = [Matrix of Trans. Probabilities][Current State]
- The predicted value is based solely on the current value

Applications of Markov Chains

- Predicting Stock Market Trends
  A hypothetical market with trends shown as below:

To / *From*	Bull	Bear	Stagnant
Bull	0.9	0.075	0.025
Bear	0.15	0.8	0.05
Stagnant	0.25	0.25	0.5

- For example, this means that the probability of going from the bull market to bear market is 0.075, but the probability of going from bear market to bull market s 0.15.

- In the previous example of mouse and maze, a state is defined as the room a mouse is in. In this problem a state is a time period. Assume in this problem a state is one week long.
- If we set the current week as bearish, then the vector of the initial state is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

  We can now calculate the probabilities of a bull, bear or a stagnant week from any number of weeks into the future.
- 1 week from now:

$$S_1 = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.8 \\ 0.05 \end{bmatrix}$$

- 5 weeks from now:

$$\begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.45 \\ 0.07 \end{bmatrix}$$

- 52 weeks from now:

$$\begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{52} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.31 \\ 0.05 \end{bmatrix}$$

- 100 weeks from now:

$$\begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{100} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.31 \\ 0.05 \end{bmatrix}$$

As the number of weeks go to infinity, the probabilities will converge to a steady state.

Hidden Markov Model (HMM)

Hidden Markov Model for Stock Trading (Nguyen, 2017)

- This is a simplified case because we know the matrix of transtion probabilities, But in many cases, we only know stock prices, but we dont know the how the market will change.
- Two stochastic processes involved:
  Observable: Stock Prices;
  Unobservable: 'State of the system'
- Basic elements of a hidden Markov model:
  Length of observation data T;
  Number of states N;
  Observation sequence O $= \{O_t, t = 1, 2...T\}$;
  Hidden state sequence Q $= \{q_t, t = 1, 2...T\}$;
  Possible values of each state $\{S_i, i = 1, 2...N\}$;
  Transition matrix $A = (a_{ij}), a_{ij} = P(q_t = S_j \mid q_{t-1} = S_{j-1})$;
  Vector of initial probability of being in state S_i at time $t = 1$:
  $p = (p_i), p_i = P(q_1 = S_i)$;
  Observation matrix B.

Hidden Markov Model (HMM)

Hidden Markov Model for Stock Trading (Nguyen, 2017)

- Data:
  SP 500 monthly prices from January 1950 to November 2016
- Observable
- Steps to predict stock prices using HMM:
  1. Choose a fixed time period T, calibrate parameters A, B, p, decide the number of states N;
  2. Move the given data (the given observation sequence) backward to get a new dataset $O^{new} = \{O_2, O_3 ... O_t - 1\}$; compute the vector $p = (p_i)$;
  3. Keep moving backward until we find a new sequence O^*, where $p_i^* = P(O^*) \approx P(O)$;
  4. Predict the price at time $T + 1$ using the formula $O_{T+1} = O_T + (O_{T^*+1} - O_{T^*}) \cdot (P(O \mid A, B) - P(O^* \mid A, B))$

Hidden Markov Model (HMM)

Hidden Markov Model for Stock Trading (Nguyen, 2017)

- Results

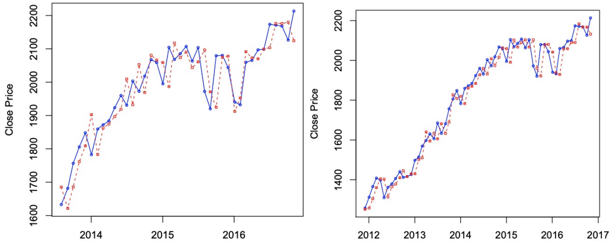

Figure A1. S&P 500 's predicted prices using the four-state HMM for 40-month out-of-sample period (**left**) and 60-month out-of-sample period (**right**).

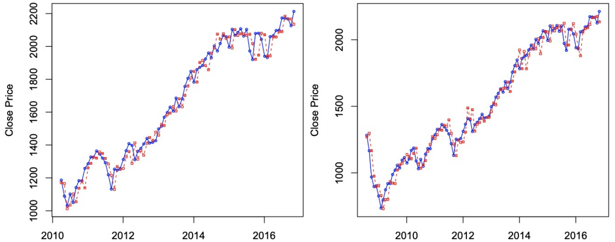

Figure A2. S&P 500 's predicted prices using the four-state HMM for 80-month out-of-sample period