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FINAN 6510
FINANCIAL APPLICATIONS OF STATISTICAL MODELS
ASSIGNMENT # 4

Exercise 1. The random walk model for a stock price assumes that at each time step the price can either increase or decrease by a fixed amount $\Delta > 0$. Suppose that P_1, $0 < P_1 < 1$ is the probability of an increase and that $P_2 = 1 - P_1$ is the probability of a decrease. Let X be the discrete random variable representing the change in a single step and Y be the discrete random variable representing the number of increases observed in 10 steps.

(a) Find the expected value of the random variable X.
(b) Find the variance of the random variable X.
(c) Describe the distribution of the random variable Y.
(d) Find the expected value of the random variable Y.

a.

$$E[X] \triangleq \sum_x x * P(X = x) = \sum_x x * f(x)$$

$$\Delta * P_1 + (-\Delta) * P_2 = \Delta(P_1 - P_2)$$

$$= \Delta\big(P_1 - (1 - P_1)\big) = \Delta(P_1 - 1 + P_1)$$

$$= \Delta(2P_1 - 1)$$

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b.

$$\bigvee[X] \triangleq E[(X - E[X])^2] = \boldsymbol{E[X^2] - (E[X])^2}$$

$$E[X] = \Delta(2P_1 - 1); E[X^2] = \Delta^2 P_1 + (-\Delta)^2(1 - P_1)$$

$$= \Delta^2 P_1 + (-\Delta)^2(1 - P_1) - [\Delta(2P_1 - 1)]^2$$

$$= \Delta^2 P_1 + \Delta^2(-P_1 + 1) - \big(\Delta(2P_1 - 1)\big)^2 = \Delta^2 P_1 + \Delta^2(-P_1 + 1) - \Delta^2(2P_1 - 1)^2$$

$$= (2P_1 + 1)^2 \colon\ 4P_1^2 + 4P_1 + 1$$

$$= \Delta^2 P_1 + \Delta^2(-P_1 + 1) - \Delta^2(4P_1^2 + 4P_1 + 1)$$

$$expand \colon \Delta^2(1 - P_1) \colon \Delta^2 - \Delta^2 P_1$$

$$= \Delta^2 P_1 + \Delta^2 - \Delta^2 P_1 - \Delta^2(4P_1^2 + 4P_1 + 1)$$

$$expand\ \ -\Delta^2(4P_1^2 + 4P_1 + 1) \colon -4\Delta^2 P_1^2 - 4\Delta^2 P_1 - \Delta^2$$

$$= \Delta^2 P_1 + \Delta^2 - \Delta^2 P_1 - 4\Delta^2 P_1^2 - 4\Delta^2 P_1 - \Delta^2$$

$$simplify \colon\ \Delta^2 P_1 + \Delta^2 - \Delta^2 P_1 - 4\Delta^2 P_1^2 + 4\Delta^2 P_1 - \Delta^2 \colon -4\Delta^2 P_1^2 + 4\Delta^2 P_1 = \boldsymbol{4\Delta^2 P_1 P_2}$$

c.

$X \sim B(P_1)$

Defining success to be an increase, so X has Bernoulli distribution with probability of success P_1.

Y is distributed as binomial with an n value of 10, because that's the number of Bernoulli trials there are and a P value of P_1, since that's the fixed probability of success across all those n equals 10 trials.

$Y \sim B(10, P_1)$

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d.

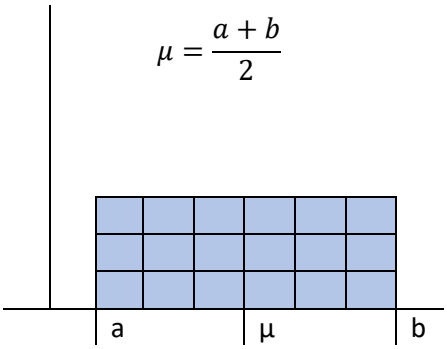
$X \sim B(P_1)$

$Y \sim B(10, P_1)$

$E[Y] = 10P_1$

Exercise 2. Let $X \sim \text{UNIF}(n)$. Verify that $\mathbb{E}[X] = \frac{a+b}{2}$.

$$E[X] \triangleq \sum_x x * P(X = x) = \sum_x x * f(x)$$



Exercise 3. Let $X \sim \text{UNIF}(n)$. Verify that $\mathbb{V}[X] = \frac{(b-a+1)^2-1}{12}$.

By definition we have: $V[X] = E[(x - E[X])^2]$

From problem #2, we found that $E[X]$, for the discrete uniform distribution, is equal to $\frac{a+b}{2}$ and $P(X) = \frac{1}{b-a+1}$ for $x \in [a, b]$. So we have: $V[X] = E\left[\left(x - \frac{a+b}{2}\right)^2\right]$

$$= E\left[x^2 - (a+b)x + \frac{(a+b)^2}{4}\right] = \sum\left(x^2 - (a+b)x + \frac{(a+b)^2}{4}\right)$$

$$= \sum\left(x^2 - (a+b)x + \frac{(a+b)^2}{4}\right) * \frac{1}{b-a+1}$$

$$= \frac{1}{b-a+1} * \left[\sum(x^2) - (a+b)\sum(x) + \frac{(a+b)^2}{4}\sum(1)\right].$$

Since each of these sums goes from $x = a\ to\ b$, we can find a formula for them as follows:

$$\sum 1 = b - a + 1$$

$$\sum x = \sum x\ (from\ 1\ to\ b) - \sum x\ \ (from\ 1\ to\ a-1)$$

$$= \frac{(b)(b+1)}{2} - \frac{(a-1)(a)}{2} = \frac{b^2 + b - a^2 + a}{2}$$

$$\sum x^2 = \sum x^2\ (from\ 1\ to\ b) - \sum x^2\ \ (from\ 1\ to\ a-1)$$

$$= \frac{b(b+1)(2b+1)}{6} - \frac{(a-1)(a)(2a-1)}{6} = \frac{2b^3 + 3b^2 + b - 2a^3 + 3a^2 - a}{6}$$

Using these, we get:

$\frac{1}{b-a+1} * [\frac{2b^3+3b^2+b-2a^3+3a^2-a}{6} - (a+b)\frac{b^2+b-a^2-a}{2} + \frac{(a+b)^2}{4}(b-a+1)]$

$= \frac{1}{12}(-2 + a - b)(a - b) = \frac{(b-a+1)^2-1}{12}$ as intended.

Exercise 4. Suppose that the daily rate of return on a certain stock is normally distributed with a mean of $\mu = 0.23\%$ and a standard deviation of $\sigma = 2.54\%$. What is the probability that the stock's price will increase by more than 5.31% on any given day next week?

$$z = \frac{x - \mu}{\sigma} = \frac{5.31 - .23}{2.54} = 2$$

$P(X > 5.31\%) = P(Z > 2) = 2.5\%$

$1 - (.95 + .025) = .025\ or\ 2.5\%$

Exercise 5. Suppose that the daily rate of return on a certain stock is normally distributed with a mean of $\mu = 0.23\%$ and a standard deviation of $\sigma = 2.54\%$.

(a) What is the probability that the stock's price will increase by more than 2% on any given day next week?

(b) What is the probability that the stock's price will decrease by more than 1% on any given day next week?

a.

$$z = \frac{x - \mu}{\sigma} = \frac{2 - .23}{2.54} = .6968$$

$P(X > 2\%) = P(Z > .6968) \approx 1 - .75705 \approx .24295$

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b.

$$z = \frac{x - \mu}{\sigma} = \frac{-1 - .23}{2.54} = -.48425197$$

$P(X < -1\%) = P(Z = -.48425197) \approx .3141$

Exercise 6. Let X be a random variable and let $c \in \mathbb{R}$ be a real number. Demonstrate that the expectation operator $\mathbb{E}$ satisfies $\mathbb{E}[cX] = c \cdot \mathbb{E}[X]$.

$$==> E[cx] = \sum_x cx * P(x) = c \sum xP(x) = cE[x] \quad \textit{The reverse direction is similar} \blacksquare$$

Exercise 7. Let X be a random variable and let $c \in \mathbb{R}$ be a real number. Demonstrate that the variance operator $\mathbb{V}$ satisfies $\mathbb{V}[cX] = c^2 \cdot \mathbb{V}[X]$.

$$==> V[cx] = E[c^2x^2] - E[cx]^2 = \sum c^2x^2p(x) - \left[\sum cxp(x)\right]^2$$
$$= c^2 \sum x^2p(x) - \left[c \sum xp(x)\right]^2 = c^2 \sum x^2p(x) - \left[c \sum xp(x)\right]^2$$
$$= c^2E[x^2p(x)] - c^2\left[E[xp(x)]\right]^2 = c^2\left[E[x^2p(x)] - \left[E[xp(x)]\right]^2\right]$$
$$= c^2[E[x^2] - E[x]^2] = c^2V(x) \quad \textit{The reverse direction is similar} \blacksquare$$

Exercise 8. The number of defaults in one year within a certain portfolio of bonds is found to be a Poisson random variable with parameter $\lambda = 7$ (i.e., the portfolio has an expectation of 7 defaults per year).

(a) Find the probability that the bond portfolio will have fewer than 3 defaults during the upcoming year.

(b) Describe the distribution of the continuous random variable representing the time between successive defaults.

(c) Find the expected time between successive defaults.

(d) Calculate the probability of there being less than 6 months between two successive defaults.

a. $= p(x < 3) = P(x = 0) + P(x = 1) + P(x = 2)$
$= \frac{7^0e^{-7}}{0!} + \frac{7^1e^{-7}}{1!} + \frac{7^2e^{-7}}{2!} = .029636$

b. Exponential; $T = Exponential(7)$

c. $E[T] = \frac{1}{\lambda} = \frac{1}{7}\ of\ one\ year; = \frac{12}{7} \approx 1.71\ months$

d. $P(T < 6) = 1 - e^{-7\frac{6}{12}} = .9698$

Exercise 9. Let X be a random variable representing a quantitative daily market dynamic (such as new information about the economy). Suppose that today's stock price S_0 for a certain company is \$120 and that tomorrow's price S_1 can be modeled by the equation $S_1 = S_0 \cdot e^X$. Assume that X is normally distributed with a mean of 0 and a variance of 0.25.

(a) Find the probability that X is less than or equal to 0.1.

(b) Suppose the daily dynamics X_i, $i = 1, 2, 3, 4$ of each of the next four consecutive days are independently and identically distributed as X so that $Y \triangleq X_1 + X_2 + X_3 + X_4$ represents the stock's four-day logarithmic return.

(i) Describe the distribution of the continuous random variable Y.

(ii) Calculate the probability that Y is less than or equal to 0.1.

a. $p(X \leq 0.1) = \int_{-\infty}^{.1} f(x)\,dx = .5793$

b. i. Sum of normal is normal distribution

ii. $Y \sim N(0 + 0 + 0 + 0, .25 + .25 + .25 + .25) = N(0,1), the\ standard\ normal\ distribution.$

So,

$$p(X \leq 0.1) = \int_{-\infty}^{.1} f(x)dx = .5398$$

(c) Describe the distribution of the random variable $\frac{S_1}{S_0}$ representing the ratio of tomorrow's price to today's price.

(d) What is the expected value of the random variable $\frac{S_1}{S_0}$?

(e) What is the variance of the random variable $\frac{S_1}{S_0}$?

c. $S_1 = S_0 e^X; \frac{S_1}{S_0} = e^X \Longrightarrow lognormal$

d. $E(x) = e^{\left(\mu + \frac{\sigma^2}{2}\right)};$

e. $V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$

Exercise 10. Let the dynamics X_i, $i = 1, 2, \ldots, d$ be independently and identically distributed as $Z \sim N(0, 1)$. One approach for modeling the short-term interest rate r_t at any time t is given by defining $r_t \triangleq X_1^2 + X_2^2 + \ldots + X_d^2$.

(a) Describe the distribution of the continuous random variable r_t.

(b) Find the probability that $r_t \in (0, 0.02]$ if $d = 3$.

(c) Find the probability that $r_t \in (0, 0.02]$ if $d = 7$.

a. *Chi squared*

b. .00075

c. 0

Exercise 11 (BONUS)**.** Prove that the variance of the sum of two independent random variables is equal to the sum of their individual variances.

$V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$

$\bigvee[X] \triangleq E[(X - E[X])^2]$

$V(X+Y) = E\left[\left((X+Y) - E(X+Y)\right)^2\right] = E[([X - E(X)] + [Y - E(Y)])^2]$

$= E\left[\left(X - E(X)\right)^2 + \left(Y - E(Y)\right)^2 + 2\left(X - E(X)\right)\left(Y - E(Y)\right)\right]$

$= E\left[\left(X - E(X)\right)^2\right] + E\left[\left(Y - E(Y)\right)^2\right] + E\left[2\left(X - E(X)\right)\left(Y - E(Y)\right)\right]$

$= V(X) + V(Y) + 2Cov(X,Y) = V(X) + V(Y) + 0 \quad as\ intended$ ■