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A Brownian Model of Financial Markets

1.1 Stocks and a Money Market

Throughout this monograph we deal with a financial market consisting of $N+1$ financial assets. One of these assets is instantaneously risk-free, and will be called a *money market.* Assets 1 through N are risky, and will be called *stocks* (although in applications of this model they are often commodities or currencies, rather than common stocks). These financial assets have continuous prices evolving continuously in time and driven by a D-dimensional Brownian motion. The continuity of the time parameter and the accompanying capacity for continuous trading permit an elegance of formulation and analysis not unlike that obtained when passing from difference to differential equations. If asset prices do not vary continuously, at least they vary frequently, and the model we propose to study has proved its usefulness as an approximation to reality. Our assumption that asset prices have no jumps is a significant one. It is tantamount to the assertion that there are no "surprises" in the market: the price of a stock at time t can be perfectly predicted from knowledge of its price at times strictly prior to t. We adopt this assumption in order to simplify the mathematics; the additional assumption that asset prices are driven by a Brownian motion is little more than a convenient way of phrasing this condition. Some literature on continuous-time markets with discontinuous asset prices is cited in the notes at the end of this chapter. The extent to which the results of this monograph can be extended to such models has not yet been fully explored.

Let us begin then with a complete probability space $(\Omega, \mathcal{F}, P)$ on which is given a standard, D-dimensional Brownian motion $W(t) = (W^{(1)}(t), \ldots, W^{(D)}(t))', 0 \leq t \leq T$. Here prime denotes transposition, so that $W(t)$ is a column vector. We assume that $W(0) = 0$ almost surely. All economic activity will be assumed to take place on a finite horizon $[0, T]$, where T is a positive constant.† Define

$$\mathcal{F}^W(t) \triangleq \sigma\{W(s); \quad 0 \leq s \leq t\}, \quad \forall t \in [0, T], \tag{1.1}$$

to be the filtration generated by $W(\cdot)$, and let $\mathcal{N}$ denote the P-null subsets of $\mathcal{F}^W(T)$. We shall use the *augmented filtration*

$$\mathcal{F}(t) \triangleq \sigma(\mathcal{F}^W(t) \cup \mathcal{N}), \quad \forall t \in [0, T]. \tag{1.2}$$

One should interpret the σ-algebra $\mathcal{F}(t)$ as the *information available to investors at time* t, in the sense that if $\omega \in \Omega$ is the true state of nature and if $A \in \mathcal{F}(t)$, then at time t all investors know whether $\omega \in A$. Note that $\mathcal{F}(0)$ contains only sets of measure one and sets of measure zero, so every $\mathcal{F}(0)$-measurable random variable is almost surely constant.

Remark 1.1: The difference between $\{\mathcal{F}^W(t)\}_{0 \leq t \leq T}$ and $\{\mathcal{F}(t)\}_{0 \leq t \leq T}$ is a purely technical one. The filtration $\{\mathcal{F}(t)\}_{0 \leq t \leq T}$ is *left-continuous*, in the sense that

$$\mathcal{F}(t) = \sigma\left(\bigcup_{0 \leq s < t} \mathcal{F}(s)\right), \quad \forall t \in (0, T], \tag{1.3}$$

and $\{\mathcal{F}^W(t)\}_{0 \leq t \leq T}$ is also left-continuous. The filtration $\{\mathcal{F}(t)\}_{0 \leq t \leq T}$ is *right-continuous* in the sense that

$$\mathcal{F}(t) = \bigcap_{t < s \leq T} \mathcal{F}(s), \quad \forall t \in [0, T), \tag{1.4}$$

but $\{\mathcal{F}^W(t)\}_{0 \leq t \leq T}$ is not right-continuous (see Karatzas and Shreve (1991), Section 2.7, for more details). Equations (1.3), (1.4) express the notion alluded to in the first paragraph of this section, that "there are no surprises in the flow of information" in this model.

Remark 1.2: Every local martingale relative to the filtration $\{\mathcal{F}(t)\}$ has a modification whose paths are *continuous* (Karatzas and Shreve (1991), Problem 3.4.16); we shall always use this continuous modification. We shall also encounter processes $Y(\cdot)$ that are right-continuous with left-hand limits and whose total variation $\check{Y}(t)$ is finite on each interval $[0, t], 0 \leq t \leq T$. We shall refer to these as *finite-variation RCLL* processes. In our context,

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†There are a few places in this book, namely, Sections 1.7, 2.6, 3.9, and 3.10, where the planning horizon is $[0, \infty)$.