Chapter 9
Optimization Problems in Finance and Insurance

We will now apply the theory of infinite horizon Markov Decision Models to solve some optimization problems in finance. In Section 9.1 we consider a *consumption and investment problem with random horizon* which leads to a contracting Markov Decision Model with infinite horizon as explained in Section 7.6.1. Explicit solutions in the case of a power utility are given. In Section 9.2 a classical *dividend pay-out problem* for an insurance company is investigated. In this example the state and action space are both discrete which implies that all functions on $E \times A$ are continuous and we can work with Theorem 7.2.1. Here the Markov Decision Model is not contracting. The main part of this section is to show that there exists an optimal stationary policy which is a so-called *band-policy.* In special cases this band-policy reduces to a barrier-policy, i.e. it is optimal to pay out all the money which is above a certain threshold. In Section 9.3 we consider a utility maximization problem in a financial market where the stock prices are *Piecewise Deterministic Markov Processes.* This optimization problem is contracting and our results from Chapters 7 and 8 allow a characterization of the value function and some computational approaches which complement the classical stochastic control approach via the *Hamilton-Jacobi-Bellman* equation. Some numerical results are also given. In Section 9.4 we study the liquidation of a large amount of shares in so-called dark pools. This is a continuous-time Markov Decision Chain with finite time horizon (see Section 8.3). Using the discrete-time solution approach we are able to derive some interesting properties of the optimal liquidation policy.

9.1 Consumption-Investment Problems with Random Horizon

In this section we reconsider the consumption and investment problem of Section 4.3. However, this time we assume that the investment horizon of the

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agent is random (cf. Section 7.6.1). This is reasonable since there may be a drastic change of the agent's plan in the future with some probability. For example the agent may need all her money because she was disabled due to an accident.
A financial market with d risky asset and one riskless bond (with interest rate $i_n = 0$) is given as introduced in Section 4.2. Recall that $\mathcal{F}_n := \mathcal{F}_n^S$. Here we assume that $R_1, R_2, \dots$ are independent and identically distributed random vectors and that the following Assumption (FM) holds:

Assumption (FM):

(i) *There are no arbitrage opportunities.*
(ii) $\mathbb{E}\,\|R_1\| < \infty$.

The first assumption means that there is no arbitrage opportunity for any finite horizon. The random horizon is here described by a *geometrically distributed* random variable τ with parameter $p \in (0,1)$, i.e.

$$\mathbb{P}(\tau = n) = (1-p)p^{n-1},\ n \in \mathbb{N}.$$

It is assumed that the random horizon τ is independent of $(\mathcal{F}_n)$. The aim is to maximize the expected discounted utility from consumption and investment until time τ. The initial wealth is given by $x > 0$. In what follows suppose that $U_c, U_p : E \to \mathbb{R}_+$ are two continuous utility functions with $dom\, U_c = dom\, U_p := [0,\infty)$ which are used to evaluate the consumption and the terminal wealth. The wealth process (X_n) evolves as follows

$$X_{n+1} = X_n - c_n + \phi_n \cdot R_{n+1},$$

where $(c,\phi) = (c_n, \phi_n)$ is a consumption-investment strategy i.e. ϕ_n and c_n are $(\mathcal{F}_n)$-adapted and $0 \le c_n \le X_n$, for all $n \in \mathbb{N}$. The optimization problem is then given by

$$\begin{cases} \mathbb{E}_x \left[\sum_{n=0}^{\tau-1} \beta^n U_c(c_n) + \beta^\tau U_p(X_\tau^{c,\phi}) \right] \to \max \\ (c,\phi) \text{ is a consumption-investment strategy and} \\ \qquad X_\tau^{c,\phi} \in dom\, U_p \ \mathbb{P}\text{-a.s.} \end{cases} \tag{9.1}$$

According to Section 7.6.1 we can formulate this optimization problem with random horizon as a stationary Markov Decision Model with infinite horizon:

- $E := [0,\infty)$ where $x \in E$ denotes the wealth,
- $A := \mathbb{R}_+ \times \mathbb{R}^d$ where $a \in \mathbb{R}^d$ is the amount of money invested in the risky assets and $c \in \mathbb{R}_+$ the amount which is consumed,
- $D(x)$ is given by

$$D(x) := \{(c,a) \in A \mid 0 \le c \le x \text{ and } x - c + a \cdot R_1 \ge 0\, \mathbb{P}\text{-a.s.}\},$$